

The first terms in the expansion of the Bergman kernel in higher degrees: mixed curvature case

Yong Wang

Abstract

We establish the cancellation of the first $|2j - q|$ terms in the diagonal asymptotic expansion of the restriction to the $(0, 2j)$ -forms of the Bergman kernel associated to the modified spin^c Dirac operator on high tensor powers of a line bundle with mixed curvature twisted by a (non necessarily holomorphic) complex vector bundle, over a compact symplectic manifold. Moreover, we give a local formula for the first and the second (non-zero) leading coefficients which generalizes the Puchol-Zhu's results.

Keywords: Modified spin^c Dirac operator; Bergman kernel; Asymptotic expansion

MSC(2010): 58J35, 32L10

1 Introduction

The study of the asymptotic expansion of Bergman kernel has attracted much attention recently. The existence of the diagonal asymptotic expansion of the Bergman kernel of high tensor powers of a positive line bundle over a compact complex manifold was first established by Tian [Ti], Ruan [Ru], Catlin [Ca], and Zelditch [Ze]. Tian [Ti], followed by Lu [LuZ] and Wang [Wa], derived explicit formulae for several terms of the asymptotic expansion on the diagonal, via Tian's method of peak sections.

Using Bismut-Lebeau's analytic localization techniques, Dai, Liu, and Ma [DLM] established the full off-diagonal asymptotic expansion of the Bergman kernel of the Spin^c Dirac operator associated with high powers of a Hermitian line bundle with positive curvature in the general context of symplectic manifolds. Moreover, they calculated the second coefficient of the expansion in the case of Kähler manifolds. Later, Ma and Marinescu [MM08] studied the expansion of generalized Bergman kernel associated with Bochner Laplacians and developed a method of formal power series to compute the coefficients. By the same method, Ma and Marinescu [MM06, Theorem 2.1] computed the second coefficient of the asymptotic expansion of the Bergman kernel of the Spin^c Dirac operator acting on high tensor powers of line bundles with positive curvature in the case of symplectic manifolds.

Recently, this asymptotic in the symplectic case found an application in the study of variation of Hodge structures of vector bundles by Charbonneau and Stern in [CS]. In their setting, the Bergman kernel is the kernel of a Kodaira-like Laplacian on a twisted bundle $L \otimes E$, where E is a (not necessarily holomorphic) complex vector bundle. Because of that, the Bergman kernel no longer concentrates in degree 0 (unlike it did in the Kähler case), and the asymptotic of its restriction to the $(0, 2j)$ -forms is related to the degree of non-holomorphicity of E . In [PZ], Puchol and Zhu showed that the leading term in the asymptotic of the restriction to the $(0, 2j)$ -forms of the Bergman kernel is of order $p^{\dim X - 2j}$ for Kähler manifolds and they computed the first and the second terms in this asymptotic. In [LuW1], Lu calculated the second coefficient of the asymptotic expansion of the Bergman kernel of the Hodge-Dolbeault operator associated with high powers of a Hermitian line bundle with non-degenerate curvature, using the method of formal power series developed by Ma and Marinescu. In this paper, we extend the Puchol-Zhu's results to the non-degenerate curvature case.

Let (X, J) be a compact connected complex manifold with complex structure J and $\dim_{\mathbb{C}} X = n$. Let (L, h^L) be a holomorphic Hermitian line bundle on X , and let ∇^L be the Chern connection of (L, h^L) with the curvature $R^L = (\nabla^L)^2$.

Our basic assumption is that $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ defines a symplectic form on X .

The complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. Since the J -invariant bilinear form $\omega(\cdot, J\cdot)$ is nondegenerate on TX , there exist J -invariant subbundles denoted $V, V^\perp \subset TX$ such that

$$\omega(\cdot, J\cdot)|_V < 0, \quad \omega(\cdot, J\cdot)|_{V^\perp} > 0 \quad (1.1)$$

and V, V^\perp are orthogonal with respect to $\omega(\cdot, J\cdot)$. Equivalently, there exist subbundles $W, W^\perp \subset T^{(1,0)}X$ such that $W \oplus W^\perp = T^{(1,0)}X$, W, W^\perp orthogonal with respect to ω and

$$R^L(u, \bar{u}) < 0, \quad \text{for } u \in W; \quad R^L(u, \bar{u}) > 0, \quad \text{for } u \in W^\perp. \quad (1.2)$$

Set $\text{rank} W = q$. Then the curvature R^L is non-degenerate of signature $(q, n - q)$. Now take the Riemannian metric g^{TX} on X to be

$$g^{TX} := -\omega(\cdot, J\cdot)|_V \oplus \omega(\cdot, J\cdot)|_{V^\perp}. \quad (1.3)$$

Since ω is compatible with the complex structure J , the metric g^{TX} is also compatible with J . Note that (X, g^{TX}) is not necessarily Kähler. Denote by $\langle \cdot, \cdot \rangle$ the \mathbb{C} -bilinear form on $TX \otimes_{\mathbb{R}} \mathbb{C}$ induced by g^{TX} . Note that $\langle \cdot, \cdot \rangle$ vanishes on $T^{(1,0)}X \times T^{(1,0)}X$ and on $T^{(0,1)}X \times T^{(0,1)}X$.

Let ∇^{TX} denote the Levi-Civita connection on X and R^{TX}, r^X denote the curvature and the scalar curvature of (TX, g^{TX}) . The metric g^{TX} induces a Hermitian metric $h^{T^{(1,0)}X}$ on $T^{(1,0)}X$ and a metric $h^{\wedge^{0,\cdot}}(T^*X) := \wedge^\cdot(T^{(0,1)}X)$. Let $\nabla^{T^{(1,0)}X}$ denote the Chern connection on $(T^{(1,0)}X, h^{T^{(1,0)}X})$ whose curvature is $R^{T^{(1,0)}X}$ and $\nabla^{T^{(1,0)}X}$ induces a Chern connection $\nabla^{\det(T^{(0,1)}X)}$ on $\det(T^{(0,1)}X) := \wedge^n T^{(0,1)}X$. Let (E, h^E) be a Hermitian complex vector bundle with a Hermitian connection ∇^E ,

whose curvature is $R^E = (\nabla^E)^2$. Let $L^p = L^{\otimes p}$ be the p^{th} tensor power of L and $\Omega^{0,\bullet}(X, L^p \otimes E) = \Gamma(X, \wedge^{0,\bullet}(T^*X) \otimes L^p \otimes E)$. We still denote by $\langle \cdot, \cdot \rangle$ be the fibre-wise metric on $\wedge^{0,\bullet}(T^*X) \otimes L^p \otimes E$ induced by g^{TX}, h^L , and h^E . Let $d\nu_X$ be the Riemannian volume of (X, g^{TX}) . The L^2 -scalar product on $\Omega^{0,\bullet}(X, L^p \otimes E)$ is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle d\nu_X(x). \quad (1.4)$$

The Chern connection $\nabla^{T^{(1,0)}X}$ on $T^{(1,0)}X$ induces naturally a Hermitian connection $\nabla^{T^{(0,1)}X}$ on $T^{(0,1)}X$. Set

$$\tilde{\nabla}^{TX} = \nabla^{T^{(1,0)}X} \oplus \nabla^{T^{(0,1)}X}. \quad (1.5)$$

Then $\tilde{\nabla}^{TX}$ is a Hermitian connection on $TX \otimes_{\mathbb{R}} \mathbb{C}$ which preserves the decomposition $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$. We still denote by $\tilde{\nabla}^{TX}$ the induced connection on TX . Let T be the torsion of the connection $\tilde{\nabla}^{TX}$, and let T_{as} be the anti-symmetrization of the tensor T , i.e., for $U, V, W \in TX$,

$$T_{as}(U, V, W) = \langle T(U, V), W \rangle + \langle T(V, W), U \rangle + \langle T(W, U), V \rangle. \quad (1.6)$$

Denote by S^B the 1-form with values in the space of antisymmetric elements of $\text{End}(TX)$ which satisfies for $U, V, W \in TX$,

$$\langle S_B(U)V, W \rangle = -\frac{1}{2}T_{as}(U, V, W). \quad (1.7)$$

Then the Bismut connection on TX is defined by

$$\nabla^B = \nabla^{TX} + S^B. \quad (1.8)$$

By [Bi, Prop. 2.5], ∇^B preserves the metric g^{TX} and the complex structure J of TX .

For any $v \in TX \otimes_{\mathbb{R}} \mathbb{C}$ with decomposition $v = v^{1,0} + v^{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\bar{v}_{1,0}^*$ be the $\langle \cdot, \cdot \rangle$ dual of $v_{1,0}$. Then $c(v) = \sqrt{2}(\bar{v}_{1,0}^* \wedge -i_{v_{0,1}})$ defines the Clifford action of v on $\wedge(T^{*(0,1)}X)$, where \wedge and i denote the standard exterior and interior multiplication, respectively. Let ∇^{Cl} denote the Clifford connection on $\wedge(T^{*(0,1)}X)$ induced canonically by ∇^{TX} and $\nabla^{\det(T^{(1,0)}X)}$. If e^1, \dots, e^{2n} denotes an orthonormal frame of T^*X , then define

$${}^c(e^{i_1} \wedge \dots \wedge e^{i_j}) = c(e_{i_1}) \cdots c(e_{i_j}), \quad \text{for } i_1 < \dots < i_j. \quad (1.9)$$

In this sense cB is defined for any $B \in \wedge^*(T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ by extending \mathbb{C} -linearity. Take $U \in TX$, Let

$$\nabla_U^{B, \wedge^{0,\bullet}} = \nabla_U^{Cl} - \frac{1}{4}{}^c(i_U T_{as}) \quad (1.10)$$

denote the Hermitian connection on $\wedge(T^{*(0,1)}X)$ induced by ∇^{Cl} and T_{as} . Then $\nabla^{B, \wedge^{0,\bullet}}$ is the Clifford connection on the spinor bundle $\wedge(T^{*(0,1)}X)$ induced by ∇^B on TX and $\nabla^{\det(T^{(1,0)}X)}$ on $\det(T^{(1,0)}X)$. The connection $\nabla^{B, \wedge^{0,\bullet}}$ preserves the Z -grading of $\wedge^{0,\bullet}(TX)$. Let w_1, \dots, w_n be an orthonormal frame of $T^{(1,0)}X$, Then

$(\overline{w_1}, \dots, \overline{w_n})$ is a local orthonormal frame of $T^{(0,1)}X$ whose dual frame is denoted by $(\overline{w^1}, \dots, \overline{w^n})$ and the vectors

$$e_{2j} = \frac{1}{\sqrt{2}}(w_j + \overline{w_j}), \quad e_{2j-1} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w_j}) \quad (1.11)$$

form a local orthonormal frame of TX . Set

$$\nabla^{TX} e_j = \Gamma^{TX} e_j, \quad \nabla^{\det(T^{(1,0)}X)}(w_1 \wedge \dots \wedge w_n) = \Gamma^{\det(T^{(1,0)}X)}(w_1 \wedge \dots \wedge w_n). \quad (1.12)$$

By [MM07, (1.3.5)], we have that $\nabla^{\wedge^{0,\bullet}}, \nabla^{B,\wedge^{0,\bullet}}$ is given, with respect to the frame $\{\overline{w^{j_1}} \wedge \dots \wedge \overline{w^{j_k}}, \quad 1 \leq j_1 < \dots < j_k \leq n\}$ of $\wedge(T^{*(0,1)}X)$, by the local formula respectively

$$\begin{aligned} & d + \frac{1}{4} \langle \Gamma^{TX} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} \Gamma^{\det(T^{(1,0)}X)}; \\ & d + \frac{1}{4} \langle \Gamma^{TX} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} \Gamma^{\det(T^{(1,0)}X)} - \frac{1}{4} c(i_{\bullet} T_{as}). \end{aligned} \quad (1.13)$$

Let $\Gamma^{B,\wedge^{0,\bullet}}$ be the connection 1-form of $\nabla^{B,\wedge^{0,\bullet}}$, i.e.,

$$\Gamma^{B,\wedge^{0,\bullet}} = \frac{1}{4} \langle \Gamma^{TX} e_i, e_j \rangle c(e_i) c(e_j) + \frac{1}{2} \Gamma^{\det(T^{(1,0)}X)} - \frac{1}{4} c(i_{\bullet} T_{as}). \quad (1.14)$$

Denote by $\nabla^{L^p \otimes E}$ the Hermitian connection on $L^p \otimes E$ induced by ∇^L and ∇^E . Set

$$\nabla^{\wedge^{0,\bullet} \otimes L^p \otimes E} = \nabla^{\wedge^{0,\bullet}} \otimes 1 + 1 \otimes \nabla^{L^p \otimes E}; \quad \nabla^{B,\wedge^{0,\bullet} \otimes L^p \otimes E} = \nabla^{B,\wedge^{0,\bullet}} \otimes 1 + 1 \otimes \nabla^{L^p \otimes E}. \quad (1.15)$$

Then $\nabla^{\wedge^{0,\bullet} \otimes L^p \otimes E}$ and $\nabla^{B,\wedge^{0,\bullet} \otimes L^p \otimes E}$ are Hermitian connections on $\wedge^{0,\bullet}(TX) \otimes L^p \otimes E$. Define the modified spin^c Dirac operator by

$$D_p^B := \sum_{j=1}^{2n} c(e_j) \nabla_{e_j}^{\wedge^{0,\bullet} \otimes L^p \otimes E} - \frac{1}{4} c(T_{as}), \quad (1.16)$$

which is a first order elliptic self-adjoint differential operator.

Definition 1.1 Let

$$P_p : \Omega^{(0,\bullet)}(X, L^p \otimes E) \rightarrow \ker(D_p^B) \quad (1.15)$$

be the orthogonal projection onto the kernel $\ker(D_p^B)$ of D_p^B . The operator P_p is called the Bergman projection. It has a smooth kernel with respect to $d\nu_X(y)$, denoted by $P_p(x, y)$, which is called the **Bergman kernel**.

We recall

Theorem 1.2 ([MM07, Thm. 8.2.4]) *There exist smooth sections \mathbf{b}_r of $\text{End}(\wedge^{0,\text{even}}(T^*X) \otimes E)$ such that for any $k \in \mathbb{N}$ and for $p \rightarrow +\infty$:*

$$p^{-n} P_p(x, x) = \sum_{r=0}^k \mathbf{b}_r(x) p^{-r} + O(p^{-k-1}), \quad (1.17)$$

that is for every $k, l \in \mathbb{N}$, there exists a constant $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$,

$$\left| p^{-n} P_p(x, x) - \sum_{r=0}^k \mathbf{b}_r(x) p^{-r} \right|_{L^l(X)} \leq C_{k,l} p^{-k-1}. \quad (1.18)$$

Here $|\cdot|_{L^l(X)}$ is the L^l -norm for the variable $x \in X$.

Let $\mathcal{R} = (R^E)^{0,2} \in \Omega^{(0,2)}(X, \text{End}(E))$ be the $(0, 2)$ -part of R^E (which is zero if E is holomorphic). Let

$$\mathcal{R} = \mathcal{R}^\top + \mathcal{R}^0 + \mathcal{R}^\perp, \quad (1.19)$$

where $\mathcal{R}^\top \in \Gamma(X, \wedge^2(\overline{W}^*) \otimes \text{End}(E))$, $\mathcal{R}^0 \in \Gamma(X, \overline{W}^* \otimes \overline{W}^{*\perp} \otimes \text{End}(E))$, $\mathcal{R}^\perp \in \Gamma(X, \wedge^2(\overline{W}^{*\perp}) \otimes \text{End}(E))$. For $1 \leq j \leq n$, let

$$I_j : \wedge^{0,\bullet}(T^*X) \otimes E \rightarrow \wedge^{0,j}(T^*X) \otimes E \quad (1.20)$$

be the natural orthogonal projection. Let $I_{\det(\overline{W}^*) \otimes E}$ denote the projection on $\det(\overline{W}^*) \otimes E$. For $j, k, l \in \mathbb{N}$, we define $B_l^{k,j}$ for $k \leq j$ by

$$B_l^{k,j} := \frac{1}{(2k+l) \times \cdots \times (2j+l)}; \quad B_l^{k,0} = 1. \quad (1.21)$$

Then we have

Theorem 1.3 For any $k \in \mathbb{N}$, $k \geq 2j$, we have when $p \rightarrow +\infty$:

$$p^{-n} I_{2j} P_p(x, x) I_{2j} = \sum_{r=2j}^k I_{2j} \mathbf{b}_{|r-q|}(x) I_{2j} p^{-|r-q|} + O(p^{-|k-q|-1}), \quad (1.22)$$

and moreover, when $2j \geq q$

$$I_{2j} \mathbf{b}_{2j-q}(x) I_{2j} = \frac{1}{(4\pi)^{2j-q}} (B_0^{1,j-\frac{q}{2}})^2 I_{2j} (\mathcal{R}_x^\perp)^{j-\frac{q}{2}} I_{\det(\overline{W}^*) \otimes E} (\mathcal{R}_x^{\perp,*})^{j-\frac{q}{2}} I_{2j}, \quad (1.23)$$

where $\mathcal{R}_x^{\perp,*}$ is the dual of \mathcal{R}_x^\perp acting on $(\wedge^{0,\bullet}(T^*X) \otimes E)_x$. When $2j < q$

$$I_{2j} \mathbf{b}_{q-2j}(x) I_{2j} = \frac{1}{(4\pi)^{q-2j}} (B_0^{1,\frac{q}{2}-j})^2 I_{2j} (\mathcal{R}_x^{\top,*})^{\frac{q}{2}-j} I_{\det(\overline{W}^*) \otimes E} (\mathcal{R}_x^\top)^{\frac{q}{2}-j} I_{2j}. \quad (1.24)$$

Let $\mathbf{J} : TX \rightarrow TX$ be the almost complex structure defined by

$$\omega(U, V) = g^{TX}(\mathbf{J}U, V) \quad \text{for } U, V \in TX. \quad (1.25)$$

Then J commutes with \mathbf{J} . Let w_1, \dots, w_n be an orthonormal frame of $(T^{(1,0)}X, h^{T^{(1,0)}X})$ such that the subbundle W is spanned by w_1, \dots, w_q , and let w^1, \dots, w^q be the dual frame. Then

$$\mathbf{J}w_j = -\sqrt{-1}w_j, \quad \text{for } j \leq q; \quad \mathbf{J}w_j = \sqrt{-1}w_j, \quad \text{for } j \geq q+1. \quad (1.26)$$

Let $T_{\mathbf{J}}^{(1,0)}X$ and $T_{\mathbf{J}}^{(0,1)}X$ be the eigenbundles of \mathbf{J} corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Set

$$u_l = \overline{w}_l \text{ if } l \leq q \text{ and } u_l = w_l \text{ otherwise.} \quad (1.27)$$

Then u_1, \dots, u_n forms an orthonormal frame of the subbundle $T_{\mathbf{J}}^{(1,0)}X$. We denote by u^1, \dots, u^n its dual frame. Then

$$\omega = \sqrt{-1} \sum_{l=1}^n u^l \wedge \overline{u}^l. \quad (1.28)$$

For $m, k, l \in \mathbb{N}$, we define $C_m(k)$ for $k \leq m$ by

$$C_m(k) := \frac{1}{(4\pi)^m} \frac{1}{2^k k!} \frac{1}{\prod_{s=k+1}^m (2s+1)}, \quad (1.29)$$

with the convention that $\prod_{s \in \emptyset} = 1$. Let $\Delta^{\wedge^{0,\bullet} \otimes E}$ be the Laplacian induced by $\nabla^{\wedge^{0,\bullet} \otimes E}$ and $\tilde{J} = -2\pi\sqrt{-1}\mathbf{J}$. Denote by $R^{\wedge^{0,\bullet}, B}$ and R^{\det} the curvature associated to the connections $\nabla^{\wedge^{0,\bullet}, B}$ and ∇^{\det} . If e_1, \dots, e_{2n} denotes an orthonormal frame of TX , then set

$$|A_1|^2 = \sum_{i < j < k} |A_1(e_i, e_j, e_k)|^2 \text{ for } A_1 \in \wedge^3(T^*X). \quad (1.30)$$

Define

$$A(Y, U, V) := \langle (\nabla_Y^{TX} \tilde{J})U, V \rangle; \quad (\nabla^X \nabla^X \tilde{J})_{(U,V)} = \nabla_U^X \nabla_V^X(\tilde{J}) - \nabla_{\nabla_U^X V}^X(\tilde{J}); \quad (1.31)$$

$$\Psi(X_0, Y, U, V) = -\frac{1}{9} \langle (\nabla_{X_0}^{TX} \tilde{J})Y, (\nabla_U^{TX} \tilde{J})V \rangle; \quad (1.32)$$

$$\begin{aligned} \widehat{\Psi}(X_0, Y, U, V) &= \frac{1}{2} \langle (\nabla^{TX} \nabla^{TX})_{(X_0, Y)} U, V \rangle \\ &+ \frac{1}{6} \left[\langle R^{TX}(X_0, V)Y, \tilde{J}U \rangle - \langle R^{TX}(X_0, U)Y, \tilde{J}V \rangle \right]; \end{aligned} \quad (1.33)$$

$$\Phi(U, V) = \frac{1}{2} \langle (\nabla^X \nabla^X \tilde{J})_{(U,V)} u_l, \overline{u}_l \rangle. \quad (1.34)$$

Write

$$\begin{aligned} A_{jl\overline{r}} &= A(u_j, u_l, \overline{u}_r); \quad A_{e_r l e_r} = A(e_r, u_l, e_r); \\ \Psi_{jl\overline{r}\overline{s}} &= \Psi(u_j, u_l, \overline{u}_r, \overline{u}_s); \quad \widehat{\Psi}_{jl\overline{r}\overline{s}} = \widehat{\Psi}(u_j, u_l, \overline{u}_r, \overline{u}_s). \end{aligned} \quad (1.35)$$

Theorem 1.4 *When $n = 4$, $j = 2$ and $q = 2$, we have*

$$I_4 \mathbf{b}_3(x) I_4 = I + II + III + IV + V, \quad (1.36)$$

where

$$I = \frac{1}{1152\pi^3} I_4 (\nabla_{\bar{u}_l}^{\wedge^0, \bullet \otimes E} \mathcal{R}_.)^\perp(x) I_{\det(\bar{W}^*) \otimes E} \left[(\nabla_{\bar{u}_l}^{\wedge^0, \bullet \otimes E} \mathcal{R}_.)^\perp(x) \right]^* I_4, \quad (1.37)$$

$$\begin{aligned} II &= \frac{1}{768\pi^3} I_4 (\Delta^{\wedge^0, \bullet \otimes E} \mathcal{R}_.)^\perp(x) I_{\det(\bar{W}^*) \otimes E} (\mathcal{R}_x^\perp)^* I_4 \\ &\quad + \frac{1}{768\pi^3} I_4 \mathcal{R}_x^\perp I_{\det(\bar{W}^*) \otimes E} (\Delta^{\wedge^0, \bullet \otimes E} \mathcal{R}_.)^{\perp,*}(x) I_4. \end{aligned} \quad (1.38)$$

III is determined by (4.11), (4.34), (4.37), (4.76), (4.79)-(4.83). IV is determined by (4.198), (4.212), (4.214), (4.215), (4.222), (4.223), (4.225), (4.226), (4.229), (4.230), (4.232). V is determined by (4.86), (4.151)-(4.153), (4.159), (4.162)-(4.172), (4.182), (4.183), (4.189), (4.194)-(4.197).

This paper is organized as follows: in Section 2 we use a local trivialization to rescale $(D_p^B)^2$, and then give the Taylor expansion of the rescaled operator. In Section 3, we use this expansion to give a formula for the coefficients \mathbf{b}_r . Finally, in Section 4, we prove Theorem 1.4. In this whole paper, when an index variables appears twice in a single term, it means that we are summing over all its possible values.

2 Rescaling $(D_p^B)^2$ and Taylor expansion

In this Section, we will give a formula for the square of $(D_p^B)^2$, then rescale the operator $(D_p^B)^2$ to get an operator \mathcal{L}_t , and give the Taylor expansion of the rescaled operator. We also study more precisely the limit operator \mathcal{L}_0 .

Let the Laplacian

$$\Delta^{B, \wedge^0, \bullet \otimes L^p \otimes E} = - \sum_{j=1}^{2n} \left[(\nabla_{e_j}^{B, \wedge^0, \bullet \otimes L^p \otimes E})^2 - \nabla_{\nabla_{e_j}^{TX} e_j}^{B, \wedge^0, \bullet \otimes L^p \otimes E} \right]. \quad (2.1)$$

Then by [MM07, Thm 1.3.7], we have

$$(D_p^B)^2 = \Delta^{B, \wedge^0, \bullet \otimes L^p \otimes E} + \frac{1}{2} p R^L(e_i, e_j) c(e_i) c(e_j) + \frac{r^X}{4} + {}^c(R^E + \frac{1}{2} R^{\det}) - \frac{1}{4} c(dT_{as}) - \frac{1}{8} |T_{as}|^2. \quad (2.2)$$

For any skew-adjoint endomorphism A of TX , it holds that

$$\begin{aligned} \frac{1}{4} \langle A e_i, e_j \rangle c(e_i) c(e_j) &= -\frac{1}{2} \langle A w_j, \bar{w}_j \rangle + \langle A w_l, \bar{w}_m \rangle \bar{w}^m \wedge i \bar{w}_l \\ &\quad + \frac{1}{2} \langle A w_l, w_m \rangle i \bar{w}_l i \bar{w}_m + \frac{1}{2} \langle A \bar{w}_l, \bar{w}_m \rangle \bar{w}^l \wedge \bar{w}^m \wedge. \end{aligned} \quad (2.3)$$

On $\Omega^{0, \bullet}(X)$, we define the operator $\omega_{d,x}$ by

$$\omega_{d,x} = -2\pi \sum_{l=1}^q i \bar{w}_l \wedge \bar{w}^l - 2\pi \sum_{l=q+1}^n \bar{w}^l \wedge i \bar{w}_l. \quad (2.4)$$

By (2.2)-(2.4) and (1.27), (1.28) and the fundamental assumption, we get

Proposition 2.1 *It holds that*

$$\begin{aligned} (D_p^B)^2 &= \triangle^{B, \wedge^{0, \bullet} \otimes L^p \otimes E} - (R^E + \frac{1}{2}R^{\det})(w_l, \bar{w}_l) + \frac{r^X}{4} - 2\pi p n - 2p\omega_d - \frac{1}{4}c(dT_{as}) \\ &- \frac{1}{8}|T_{as}|^2 + 2(R^E + \frac{1}{2}R^{\det})(w_l, \bar{w}_m)\bar{w}^m \wedge i_{\bar{w}_l} + R^E(w_l, w_m)i_{\bar{w}_l}i_{\bar{w}_m} + R^E(\bar{w}_l, \bar{w}_m)\bar{w}^l \wedge \bar{w}^m \wedge. \end{aligned} \quad (2.5)$$

Fix $x_0 \in X$ and w_j an orthonormal basis of $T_{x_0}^{(1,0)}X$, with dual basis w^j , and we construct an orthonormal basis $\{e_l\}$ of $T_{x_0}X$ from $\{w_j\}$ as in (1.11). For $\varepsilon > 0$, we denote by $B^X(x_0, \varepsilon)$ and $B^{T_{x_0}X}(0, \varepsilon)$ the open balls in X and $T_{x_0}X$ with center x_0 and 0 and radius ε respectively. If $\exp_{x_0}^X$ is the Riemannian exponential of X , then for ε small enough, $Z \in B^{T_{x_0}X}(0, \varepsilon) \rightarrow \exp_{x_0}^X(Z) \in B^X(x_0, \varepsilon)$ is a diffeomorphism, which gives local coordinates by identifying $T_{x_0}X$ with \mathbb{R}^{2n} via the orthonormal basis $\{e_l\}$:

$$(Z_1, \dots, Z_{2n}) \in \mathbb{R}^{2n} \mapsto \sum_l Z_l e_l \in T_{x_0}X. \quad (2.6)$$

From now on, we will always identify $B^{T_{x_0}X}(0, \varepsilon)$ and $B^X(x_0, \varepsilon)$. Note that in this identification, the radial vector field $\mathcal{R} = \sum_l Z_l e_l$ becomes $\mathcal{R} = Z$, so Z can be viewed as a point or as a tangent vector.

For $Z \in B^{T_{x_0}X}(0, \varepsilon)$, we identify (L_Z, h_Z^L) , (E_Z, h_Z^E) and $(\wedge^{0, \bullet}(T^*X)_Z, h_Z^{\wedge^{0, \bullet}})$ with $(L_{x_0}, h_{x_0}^L)$, $(E_{x_0}, h_{x_0}^E)$ and $(\wedge^{0, \bullet}(T^*X)_{x_0}, h_{x_0}^{\wedge^{0, \bullet}})$ by parallel transport with respect to the connection ∇^L , ∇^E and $\nabla^{\wedge^{0, \bullet}, B}$. Let S_L be a unit vector of L_{x_0} . It gives an isometry $L_{x_0}^p \simeq \mathbb{C}$, which yields to an isometry $\mathbb{E}_p := \wedge^{0, \bullet}(T^*X) \otimes L^p \otimes E \simeq \wedge^{0, \bullet}(T^*X) \otimes E$. Let $d\nu_{TX}$ be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$, and $\kappa(Z)$ be the smooth positive function defined for $|Z| \leq \varepsilon$ by

$$d\nu_X(Z) = \kappa(Z)d\nu_{TX}(Z), \quad (2.7)$$

with $\kappa(0) = 1$.

Definition 2.2 We denote by ∇_U the ordinary differentiation operator in the direction U on $T_{x_0}X$. For $s \in \Gamma(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$, and for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned} (S_t s)(Z) &= s(Z/t), \\ \nabla_t^B &= t S_t^{-1} \kappa^{\frac{1}{2}} \nabla^{Cl_0, B} \kappa^{-\frac{1}{2}} S_t, \\ \nabla_{0, \cdot} &= \nabla \cdot + \frac{1}{2} R_{x_0}^L(Z, \cdot), \\ \mathcal{L}_t &= t^2 S_t^{-1} \kappa^{\frac{1}{2}} (D_p^B)^2 \kappa^{-\frac{1}{2}} S_t, \\ \mathcal{L}_0 &= -(\nabla_{0, e_j})^2 - 2n\pi - 2\omega_{d, x_0}. \end{aligned} \quad (2.8)$$

Let $\|\cdot\|_{L^2}$ be the L^2 -norm induced by $h^{\mathbb{E}_{x_0}}$ and $d\nu_{TX}$, we have

Theorem 2.3 *There exist second order formally self-adjoint (with respect to $\|\cdot\|_{L^2}$) differential operators \mathcal{O}_r with polynomial coefficients such that for all $m \in \mathbb{N}$,*

$$\mathcal{L}_t = \mathcal{L}_0 + \sum_{r=1}^m t^r \mathcal{O}_r + O(t^{m+1}) \quad (2.9)$$

Furthermore, each \mathcal{O}_r can be decomposed as

$$\mathcal{O}_r = \mathcal{O}_r^0 + \mathcal{O}_r^{+2} + \mathcal{O}_r^{-2}, \quad (2.10)$$

where \mathcal{O}_r^k changes the degree of the form which it acts on by k .

Proof. By (5.22) in [LuW1], we have

$$\begin{aligned} \frac{1}{4}c(dT_{as}) &= \frac{1}{8}dT_{as}(w_i, \bar{w}_i, w_j, \bar{w}_j) - \frac{1}{2}dT_{as}(w_i, \bar{w}_j, w_k, \bar{w}_k)\bar{w}^j \wedge i\bar{w}_i \\ &\quad + \frac{1}{4}dT_{as}(w_i, w_j, \bar{w}_k, \bar{w}_l)\bar{w}^k \wedge \bar{w}^l \wedge i\bar{w}_i i\bar{w}_j. \end{aligned} \quad (2.11)$$

So $\frac{1}{4}c(dT_{as})$ preserves the degree of $\wedge^{0,\bullet}(T^*X)$. By $\nabla^B(J) = 0$, we know that $\nabla^{Cl,B}$ also preserves the degree of $\wedge^{0,\bullet}(T^*X)$. By Proposition 2.1, similar to the proof of Theorem 1.8 in [PZ], we can prove this theorem. \square

Recall Theorem 3.5.1 in [LuW2], we may get

Theorem 2.4 *It holds that*

$$\mathcal{O}_1 = -\frac{2}{3}(\partial_s R^L)_{x_0}(\mathcal{R}, e_i)Z_s \nabla_{0,e_i} - \frac{1}{3}(\partial_s R^L)_{x_0}(\mathcal{R}, e_s) - \pi\sqrt{-1} \langle (\nabla_{\mathcal{R}}^B \mathbf{J})e_i, e_l \rangle c(e_i)c(e_l), \quad (2.12)$$

$$\begin{aligned} \mathcal{O}_2^0 &= \frac{1}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_s \rangle \nabla_{0,e_i} \nabla_{0,e_s} + \left[\frac{2}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_s)e_s, e_i \rangle \right. \\ &\quad \left. - \left(\frac{1}{2} \sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} + R_{x_0}^E + R_{x_0}^{\wedge^{0,\bullet},B} \right) (\mathcal{R}, e_i) \right] \nabla_{0,e_i} \\ &\quad - \frac{1}{4} \nabla_{e_s} \left(\sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0}(\mathcal{R}, e_s) \frac{Z^\alpha}{\alpha!} \right) \\ &\quad - \frac{1}{9} \sum_i \left[\sum_s (\partial_s R^L)_{x_0}(\mathcal{R}, e_i) Z_s \right]^2 \\ &\quad + \left[- (R^E + \frac{1}{2}R^{\det})(w_l, \bar{w}_l) + \frac{r^X}{4} - \frac{1}{4}c(dT_{as}) \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8}|T_{as}|^2 + 2(R^E + \frac{1}{2}R^{\det})(w_l, \bar{w}_m)\bar{w}^m \wedge i\bar{w}_l \Big] (x_0) \\
& + \frac{1}{12} \left[\sum_s (\nabla_{0,e_s})^2, \langle R_{x_0}^{TX}(\mathcal{R}, e_l)\mathcal{R}, e_l \rangle \right] \\
& - \frac{\pi}{2}\sqrt{-1} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} e_i, e_l \rangle c(e_i)c(e_l)
\end{aligned} \tag{2.13}$$

$$\mathcal{O}_2^{+2} = \mathcal{R}_{x_0}, \quad \mathcal{O}_2^{-2} = (\mathcal{R}_{x_0})^*. \tag{2.14}$$

In the following, we will study the limit operator \mathcal{L}_0 . We introduce the complex coordinates $\xi = (\xi_1, \dots, \xi_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. We get $Z = \xi + \bar{\xi}$, $w_j = \sqrt{2}\frac{\partial}{\partial \xi_j}$ and $\bar{w}_j = \sqrt{2}\frac{\partial}{\partial \bar{\xi}_j}$. We will identify ξ to $\sum_j \xi_j \frac{\partial}{\partial \xi_j}$ and $\bar{\xi}$ to $\sum_j \bar{\xi}_j \frac{\partial}{\partial \bar{\xi}_j}$ when we consider ξ and $\bar{\xi}$ as vector fields. Set

$$z = (\bar{\xi}_1, \dots, \bar{\xi}_q, \xi_{q+1}, \dots, \xi_n), \quad \bar{z} = (\xi_1, \dots, \xi_q, \bar{\xi}_{q+1}, \dots, \bar{\xi}_n). \tag{2.15}$$

Then

$$\mathbf{J}\partial_{z_l} = \sqrt{-1}\partial_{z_l}, \quad \mathbf{J}\partial_{\bar{z}_l} = -\sqrt{-1}\partial_{\bar{z}_l}, \quad \text{for } l = 1, \dots, n. \tag{2.16}$$

We will identify z to $\sum_j z_j \frac{\partial}{\partial z_j}$ and \bar{z} to $\sum_j \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$ when we consider z and \bar{z} as vector fields. Then $Z = \xi + \bar{\xi} = z + \bar{z}$. Set $u_j = \sqrt{2}\partial_{z_j}$ and

$$f_{2j} = \frac{1}{\sqrt{2}}(u_j + \bar{u}_j), \quad f_{2j-1} = \frac{\sqrt{-1}}{\sqrt{2}}(u_j - \bar{u}_j). \tag{2.17}$$

Then $\{u_1, \dots, u_n\}$ forms an orthonormal basis of $T_{\mathbf{J}, x_0}^{(1,0)}X$ and f_1, \dots, f_{2n} is an orthonormal basis of $T_{x_0}X$. Set

$$\begin{aligned}
b_i &= -2\nabla_{0, \frac{\partial}{\partial z_i}}, \quad b_i^+ = 2\nabla_{0, \frac{\partial}{\partial \bar{z}_i}}, \\
b &= (b_1, \dots, b_n), \quad \mathcal{L} = -\sum_i (\nabla_{0, e_i})^2 - 2\pi n.
\end{aligned} \tag{2.18}$$

By definition, $\nabla_0 = \nabla + \frac{1}{2}R_{x_0}^L(Z, \cdot)$, so we get

$$b_i = -2\frac{\partial}{\partial z_i} + \pi\bar{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial \bar{z}_i} + \pi z_i, \tag{2.19}$$

and for any polynomial $g(z, \bar{z})$ in z and \bar{z} ,

$$\begin{aligned}
[b_i, b_j^+] &= -4\pi\delta_{ij}, \quad [b_i, b_j] = [b_i^+, b_j^+] = 0, \\
[g(z, \bar{z}), b_j] &= 2\frac{\partial}{\partial z_j}g(z, \bar{z}), \quad [g(z, \bar{z}), b_j^+] = -2\frac{\partial}{\partial \bar{z}_j}g(z, \bar{z}).
\end{aligned} \tag{2.20}$$

and

$$\mathcal{L} = \sum_{l=1}^n b_l b_l^+, \quad \mathcal{L}_0 = \mathcal{L} - 2\omega_{d,x_0}. \quad (2.21)$$

Then $b_l^+ = (b_l)^*$, and \mathcal{L} , \mathcal{L}_0 are self-adjoint with respect to the L^2 norm. We recall

Theorem 2.5 ([MM07, Thm. 8.2.3]) *The spectrum of the restriction of \mathcal{L} to $L^2(\mathbb{R}^{2n})$ is $\text{Sp}(\mathcal{L}|_{L^2(\mathbb{R}^{2n})}) = 4\pi\mathbb{N}$ and an orthogonal basis of the eigenspace for the eigenvalue $4\pi k$ is*

$$b^\alpha \left(z^\beta \exp \left(-\frac{\pi}{2} |z|^2 \right) \right), \quad \text{with } \alpha, \beta \in \mathbb{N}^n \text{ and } \sum_i \alpha_i = k. \quad (2.22)$$

Especially, an orthonormal basis of $\ker(\mathcal{L}|_{L^2(\mathbb{R}^{2n})})$ is

$$\left(\frac{\pi^{|\beta|}}{\beta!} \right)^{\frac{1}{2}} z^\beta \exp \left(-\frac{\pi}{2} |z|^2 \right), \quad (2.23)$$

and thus if $\mathcal{P}(Z, Z')$ is the smooth kernel of \mathcal{P} the orthogonal projection from $(L^2(\mathbb{R}^{2n}), \|\cdot\|_0)$ onto $\ker(\mathcal{L})$ (where $\|\cdot\|_0$ is the L^2 -norm associated to $g_{x_0}^{TX}$) with respect to $d\nu_{TX}(Z')$, we have

$$\mathcal{P}(Z, Z) = \exp \left(-\frac{\pi}{2} (|z|^2 + |z'|^2 - 2zz') \right). \quad (2.24)$$

Now let P^N be the orthogonal projection from $(L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0}), \|\cdot\|_{L^2})$ onto $N := \ker(\mathcal{L}_0)$, and $P^N(Z, Z')$ be its smooth kernel with respect to $d\nu_{TX}(Z')$. From (2.21), we have:

$$P^N(Z, Z') = \mathcal{P}(Z, Z') I_{\det(\overline{W}^*) \otimes E}. \quad (2.25)$$

3 The first coefficient in the asymptotic expansion

In this section, we will compute the first coefficient in the asymptotic expansion. By Theorem 2.5 and (2.21), we get that for every $\lambda \in S^1$ the unit circle in \mathbb{C} , $(\lambda - \mathcal{L}_0)^{-1}$ exists. Let $f(\lambda, t)$ be a formal power series on t with values in $\text{End}(L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0}))$:

$$f(\lambda, t) = \sum_{r=0}^{+\infty} t^r f_r(\lambda) \quad \text{with } f_r(\lambda) \in \text{End}(L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})). \quad (3.1)$$

Consider the equation of formal power series on t for $\lambda \in S^1$:

$$\left(\lambda - \mathcal{L}_0 - \sum_{r=1}^{+\infty} t^r \mathcal{O}_r \right) f(\lambda, t) = \text{Id}_{L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})}. \quad (3.2)$$

By induction, we get

$$f_r(\lambda) = \left(\sum_{\substack{r_1 + \dots + r_k = r \\ r_j \geq 1}} (\lambda - \mathcal{L}_0)^{-1} \mathcal{O}_{r_1} \dots (\lambda - \mathcal{L}_0)^{-1} \mathcal{O}_{r_k} \right) (\lambda - \mathcal{L}_0)^{-1}. \quad (3.3)$$

We define \mathcal{F}_r by

$$\mathcal{F}_r = \frac{1}{2\pi\sqrt{-1}} \int_{S^1} f_r(\lambda) d\lambda, \quad (3.4)$$

and we denote by $\mathcal{F}_r(Z, Z')$ its smooth kernel with respect to $d\nu_{TX}(Z')$.

Theorem 3.1 ([MM07, Thm 8.2.4]) *The following equation holds*

$$\mathbf{b}_r(x_0) = \mathcal{F}_{2r}(0, 0). \quad (3.5)$$

Proof of Theorem 1.3. Let $T_{\mathbf{r}}(\lambda) = (\lambda - \mathcal{L}_0)^{-1} \mathcal{O}_{r_1} \dots (\lambda - \mathcal{L}_0)^{-1} \mathcal{O}_{r_k} (\lambda - \mathcal{L}_0)^{-1}$ be the term in the sum (3.3) corresponding to $\mathbf{r} = (r_1, \dots, r_k)$. Let N^\perp be the orthogonal of N in $L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$, and P^{N^\perp} be the associated orthogonal projector. In $T_{\mathbf{r}}(\lambda)$, each term $(\lambda - \mathcal{L}_0)^{-1}$ can be decomposed as

$$(\lambda - \mathcal{L}_0)^{-1} = (\lambda - \mathcal{L}_0)^{-1} P^{N^\perp} + \frac{1}{\lambda} P^N. \quad (3.6)$$

Set

$$L^{N^\perp}(\lambda) = (\lambda - \mathcal{L}_0)^{-1} P^{N^\perp}, \quad L^N(\lambda) = \frac{1}{\lambda} P^N. \quad (3.7)$$

Then $(\lambda - \mathcal{L}_0)^{-1}$, $L^{N^\perp}(\lambda)$, $L^N(\lambda)$ preserve the degree. For $\eta = (\eta_1, \dots, \eta_{k+1}) \in \{N, N^\perp\}^{k+1}$, let

$$T_{\mathbf{r}}^\eta(\lambda) = L^{\eta_1}(\lambda) \mathcal{O}_{r_1} \dots L^{\eta_k}(\lambda) \mathcal{O}_{r_k} L^{\eta_{k+1}}(\lambda). \quad (3.8)$$

So we have

$$T_{\mathbf{r}}(\lambda) = \sum_{\eta=(\eta_1, \dots, \eta_{k+1})} T_{\mathbf{r}}^\eta(\lambda), \quad (3.9)$$

$$\mathcal{F}_{2r} = \frac{1}{2\pi\sqrt{-1}} \sum_{\substack{|\mathbf{r}| = 2r, \eta \\ r_j \geq 1}} \int_{S^1} T_{\mathbf{r}}^\eta(\lambda) d\lambda. \quad (3.10)$$

Since $L^{N^\perp}(\lambda)$ is a holomorphic function of λ , so in (3.10), every non-zero term that appears at least one $L^N(\lambda)$, that is there exists i_0 such that $\eta_{i_0} = N$. Now fix k and j in \mathbb{N} . Let $s \in L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$ be a form of degree $2j$, $\mathbf{r} \in (\mathbb{N}/0)^k$ such that $\sum_i r_i = 2r$ and $\eta = (\eta_1, \dots, \eta_{k+1}) \in \{N, N^\perp\}^{k+1}$ such that there is a i_0 satisfying $\eta_{i_0} = N$. We

want to find a necessary condition for $I_{2j}T_{\mathbf{r}}^\eta(\lambda)I_{2j}$ to be non-zero.

Case I) all $r_l \geq 2$. Since $L^{\eta_{i_0}} = \frac{1}{\lambda}P^N$, and N is concentrated in degree q , we must have that

$$\deg(\mathcal{O}_{r_{i_0}}L^{\eta_{i_0}+1}(\lambda)\mathcal{O}_{r_{i_0}+1}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s)$$

has the degree q component. But each $L^\eta(\lambda)$ preserves the degree and \mathcal{O}_{r_i} for $r_i \geq 2$ rises or lowers the degree at most by 2, so

$$q \geq 2j - 2(k - i_0 + 1), \quad 2j + 2(k - i_0 + 1) \geq q. \quad (3.11)$$

Similarly, $L^{\eta_1}(\lambda)\mathcal{O}_{r_1}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s$ must have a non-zero component in degree $2j$, then

$$2j \leq 2(i_0 - 1) + q, \quad q - 2(i_0 - 1) \leq 2j. \quad (3.12)$$

By (3.11) and (3.12), we have

$$4j \leq 2q + 2k, \quad 2q - 4j \leq 2k. \quad (3.13)$$

By $r_l \geq 2$, then $2k \leq 2r$. So

$$|2j - q| \leq r. \quad (3.14)$$

Case II) at least one $r_j = 1$. We assume that $l \geq 1$ and there are l_1 terms $\mathcal{O}_{r_\alpha} = \mathcal{O}_1$ for $k \geq \alpha \geq i_0$ and there are $l - l_1$ terms $\mathcal{O}_{r_\beta} = \mathcal{O}_1$ for $1 \leq \beta \leq i_0 - 1$. Suppose that $I_{2j}T_{\mathbf{r}}^\eta(\lambda)I_{2j} \neq 0$. Since $L^{\eta_{i_0}} = \frac{1}{\gamma}P^N$, N is concentrated in degree q , we must have

$$\deg(\mathcal{O}_{r_{i_0}}L^{\eta_{i_0}+1}(\lambda)\mathcal{O}_{r_{i_0}+1}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s) = q \quad (3.15)$$

We know that $L^{\eta_i}(\lambda)$ and \mathcal{O}_1 preserve the degree, and \mathcal{O}_{r_i} for $r_i \geq 2$ rises or lowers the degree at most by 2, so

$$q \geq 2j - 2(k - i_0 + 1 - l_1), \quad q \leq 2j + 2(k - i_0 + 1 - l_1). \quad (3.16)$$

Similarly, $L^{\eta_1}(\lambda)\mathcal{O}_{r_1}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s$ must have a non-zero component in degree $2j$ and \mathcal{O}_{r_i} for $r_i \geq 2$ rises or lowers the degree at most by 2, so we have

$$2j \leq q + 2[i_0 - 1 - (l - l_1)], \quad 2j \geq q - 2[i_0 - 1 - (l - l_1)] \quad (3.17)$$

By (3.16) and (3.17), we get $4j \leq 2q + 2k - 2l$ and $2q \leq 4j + 2k - 2l$. Finally, since $2r = \sum_{i=1}^k r_i \geq 2(k - l) + l = 2k - l$, we have $2k - l \leq 2r$. So

$$4j \leq 2q + 2r - l, \quad 2q \leq 4j + 2r - l. \quad (3.18)$$

By $1 \leq l$, then

$$|2j - q| + \frac{1}{2} \leq |2j - q| + \frac{l}{2} \leq r. \quad (3.19)$$

Consequently, if $r < |2j - q|$, we have $I_{2j}T_{\mathbf{r}}^\eta(\lambda)I_{2j} = 0$, and if $r = |2j - q|$, only terms in case I) contribute to $I_{2j}T_{\mathbf{r}}^\eta(\lambda)I_{2j}$.

For the second part of this theorem, let us assume that we are in the limit case where $r = |2j - q|$. When $2j \geq q$, by (3.13) and $2k \leq 2r = 4j - 2q$, we get $k = 2j - q$ and $r = k$. By $r_i \geq 2$, then $r_i = 2$ for any i . By (3.11) and (3.12), then

$$i_0 = \frac{k}{2} + 1 = j + 1 - \frac{q}{2}, \quad (3.20)$$

and similar to (2.18) in [PZ], we have

$$I_{2j}\mathcal{F}_{2(2j-q)}I_{2j} = I_{2j}(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}}P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})^{j-\frac{q}{2}}I_{2j}. \quad (3.21)$$

Let $A = I_{2j}(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}}P^N$, then

$$I_{2j}\mathcal{F}_{2(2j-q)}I_{2j} = AA^*. \quad (3.22)$$

By (2.4), (2.21) and (2.25), similar to (2.20) in [PZ], we have

$$A = \frac{1}{(4\pi)^{j-\frac{q}{2}}}B_0^{1,j-\frac{q}{2}}I_{2j}(\mathcal{R}_{x_0}^\perp)^{j-\frac{q}{2}}P^N. \quad (3.23)$$

By (2.21) in [PZ] and (3.22), (3.23), we get (1.23). Using the same method, we have (1.24) when $2j < q$. \square

4 The second coefficient in the asymptotic expansion

In this section, we prove Theorem 1.4. Using (3.5), we know that

$$I_{2j}\mathbf{b}_{2j+1-q}I_{2j}(0,0) = I_{2j}\mathcal{F}_{4j+2-2q}I_{2j}(0,0). \quad (4.1)$$

In Section 4.1, we decompose this term into 5 terms, and then in Sections 4.2, 4.3 and 4.4, we handle them separately. For the rest of the section we fix an integer $j \in [0, n]$. For every smoothing operator F acting on $L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$ that appears in this section, we will denote by $F(Z, Z')$ its smooth kernel with respect to $d\nu_{TX}(Z')$.

4.1. Decomposition of the problem.

For $r = 2j + 1 - q$, using the same discussions in Section 3 in [PZ], we see that in $I_{2j}\mathcal{F}_{4j+2-2q}I_{2j}(0,0)$ there are 3 types of terms $T_{\mathbf{b}}^\eta(\lambda)$ from case I) with non-zero integral, in which:

- for $k = 2j - q$:
 - there are $2j - 2 - q$ \mathcal{O}_{r_i} equal to \mathcal{O}_2 and 2 equal to \mathcal{O}_3 : we will denote by I the sum of these terms,
 - there are $2j - 1 - q$ \mathcal{O}_{r_i} equal to \mathcal{O}_2 and 1 equal to \mathcal{O}_4 : we will denote by II the sum of these terms,
- for $k = 2j + 1 - q$:

-all the \mathcal{O}_{r_i} are equal to \mathcal{O}_2 : we will denote by III the sum of these terms.
For $r = 2j + 1 - q$, by (3.18), we have

$$4j \leq 2q + 2k - 2l \leq 2q + 2r - l = 4j + 2 - l, \quad (4.2)$$

then $1 \leq l \leq 2$, and $l = 1$ or 2 .

For $l = 1$, then $4j \leq 2q + 2k - 2 \leq 4j + 1$, so $k = 2j + 1 - q$. By $4j + 2 - 2q = 2r = \sum_{i=1}^k r_i$ and $l = 1$, so there are $2j - 1 - q$ \mathcal{O}_{r_i} equal to \mathcal{O}_2 and 1 equal to \mathcal{O}_3 and 1 equal to \mathcal{O}_1 ; we will denote by IV the sum of these terms.

For $l = 2$, then $4j \leq 2q + 2k - 4 \leq 4j$, so $k = 2j + 2 - q$. By $4j + 2 - 2q = 2r = \sum_{s=1}^k r_s$ and $l = 2$, so there are $2j - q$ \mathcal{O}_{r_i} equal to \mathcal{O}_2 and 2 equal to \mathcal{O}_1 ; we will denote by V the sum of these terms.

We have a decomposition

$$I_{2j} \mathcal{F}_{4j+2-2q} I_{2j}(0, 0) = \text{I} + \text{II} + \text{III} + \text{IV} + \text{V}. \quad (4.3)$$

Similar to Remark 3.1 in [PZ], we know that only $\mathcal{O}_2^{\pm 2}$, $\mathcal{O}_3^{\pm 2}$ and $\mathcal{O}_4^{\pm 2}$ in I and II, and not some $\mathcal{O}_{r_i}^0$. So by (0.13), (0.14) and (0.15) in [PZ], we get cases I, II and we only need change j to $j - \frac{q}{2}$ and change \mathcal{R} to \mathcal{R}^\perp . Then we have

$$\text{I} = \text{I}_a + \text{I}_a^* + \text{I}_b, \quad (4.4)$$

if $j - \frac{q}{2} = 0, 1$, then $\text{I}_a = 0$, and if $j - \frac{q}{2} \geq 2$,

$$\begin{aligned} \text{I}_a = & \frac{C_{j-\frac{q}{2}}(j - \frac{q}{2})}{2\pi} I_{2j} \sum_{\alpha=0}^{j-\frac{q}{2}-2} \sum_{s=0}^{\alpha} \left\{ (C_{j-\frac{q}{2}}(j - \frac{q}{2}) - C_{j-\frac{q}{2}}(\alpha + 1)) \right. \\ & \cdot \mathcal{R}_x^{\perp j-\frac{q}{2}-(\alpha+2)} (\nabla_{\overline{u_l}}^{\wedge 0, \bullet \otimes E} \mathcal{R}_\cdot)^\perp(x) \mathcal{R}_x^{\perp \alpha-s} (\nabla_{u_l}^{\wedge 0, \bullet \otimes E} \mathcal{R}_\cdot)^\perp(x) \mathcal{R}_x^{\perp s} \\ & + C_{j-\frac{q}{2}}(s) \left[\prod_{s_1=\alpha+2}^{j-\frac{q}{2}} (1 + \frac{1}{2s_1}) - 1 \right] \mathcal{R}_x^{\perp j-\frac{q}{2}-(\alpha+2)} (\nabla_{u_l}^{\wedge 0, \bullet \otimes E} \mathcal{R}_\cdot)^\perp(x) \\ & \left. \mathcal{R}_x^{\perp \alpha-s} (\nabla_{\overline{u_l}}^{\wedge 0, \bullet \otimes E} \mathcal{R}_\cdot)^\perp(x) \mathcal{R}_x^{\perp s} \right\} I_{\det(\overline{W}^*) \otimes E} (\mathcal{R}_x^{\perp j-\frac{q}{2}})^* I_{2j}, \end{aligned} \quad (4.5)$$

if $j - \frac{q}{2} = 0$, then $\text{I}_b = 0$, and if $j - \frac{q}{2} \geq 1$,

$$\begin{aligned} \text{I}_b = & \frac{1}{2\pi} I_{2j} \left[\sum_{k=0}^{j-\frac{q}{2}-1} (C_{j-\frac{q}{2}}(j - \frac{q}{2}) - C_{j-\frac{q}{2}}(k)) (\mathcal{R}_x^\perp)^{j-\frac{q}{2}-k-1} (\nabla_{\overline{u_l}}^{\wedge 0, \bullet \otimes E} \mathcal{R}_\cdot)^\perp(x) \mathcal{R}_x^{\perp k} \right] \\ & \times I_{\det(\overline{W}^*) \otimes E} \left[\sum_{k=0}^{j-\frac{q}{2}-1} (C_{j-\frac{q}{2}}(j - \frac{q}{2}) - C_{j-\frac{q}{2}}(k)) \mathcal{R}_x^{\perp j-\frac{q}{2}-k-1} (\nabla_{u_l}^{\wedge 0, \bullet \otimes E} \mathcal{R}_\cdot)^\perp(x) \mathcal{R}_x^{\perp k} \right]^* I_{2j}, \end{aligned} \quad (4.6)$$

$$\text{II} = \text{II}_a + \text{II}_a^*, \quad (4.7)$$

if $j - \frac{q}{2} = 0$, then $\Pi_a = 0$, and if $j - \frac{q}{2} \geq 1$,

$$\begin{aligned} \Pi_a &= \frac{C_{j-\frac{q}{2}}(j - \frac{q}{2})}{4\pi} I_{2j} \sum_{k=0}^{j-\frac{q}{2}-1} \left\{ (C_{j-\frac{q}{2}}(j - \frac{q}{2}) - C_{j-\frac{q}{2}}(k)) \right. \\ &\quad \cdot (\mathcal{R}_x^\perp)_x^{j-\frac{q}{2}-(k+1)} (\Delta^{\wedge^{0,\bullet} \otimes E} \mathcal{R}_\cdot)^\perp(x) (\mathcal{R}_x^\perp)^k \Big\} I_{\det(\overline{W}^*) \otimes E} ((\mathcal{R}_x^\perp)^{j-\frac{q}{2}})^* I_{2j}. \end{aligned} \quad (4.8)$$

4.2 The term involving only \mathcal{O}_2

Lemma 4.1 *Any term $T_{\mathbf{r}}^\eta$ appearing in the term III (with non-vanishing integral) has three types, III_0 : terms with $\eta_{j-\frac{q}{2}+1} = \eta_{j-\frac{q}{2}+2} = N$ and other $\eta_i = N^\perp$, III_a : terms with $\eta_{j-\frac{q}{2}+1} = N$ and other $\eta_i = N^\perp$, III_b : terms with $\eta_{j-\frac{q}{2}+2} = N$ and other $\eta_i = N^\perp$. We have:*

$$\begin{aligned} \text{III}_0 &= \sum_{l=1}^{j-\frac{q}{2}} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{l-1} (\mathcal{L}_0^{-2} \mathcal{O}_2^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-l} P^N \mathcal{O}_2^0 P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}} I_{2j} \\ &+ \sum_{l=1}^{j-\frac{q}{2}} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N \mathcal{O}_2^0 P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{l-1} (\mathcal{O}_2^{-2} \mathcal{L}_0^{-2}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-l} I_{2j}, \end{aligned} \quad (4.9)$$

$$\text{III}_a = \sum_{k=0}^{j-\frac{q}{2}} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} (\mathcal{L}_0^{-1} \mathcal{O}_2^0) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^k P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}} I_{2j}, \quad (4.10)$$

$$\text{III}_b = (\text{III}_a)^*, \quad \text{III} = \text{III}_0 + \text{III}_a + \text{III}_b. \quad (4.11)$$

Proof. Fix a term $T_{\mathbf{r}}^\eta$ appearing in the term III with non-vanishing integral. Using again the same reasoning as in Section 2.2 in [PZ], we see that there exists at most two indices i_0 such that $\eta_{i_0} = N$, and that they are in $\{j - \frac{q}{2} + 1, j - \frac{q}{2} + 2\}$. Now, the only possible term with $\eta_{j-\frac{q}{2}+1} = \eta_{j-\frac{q}{2}+2} = N$ is:

$$(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N \mathcal{O}_2^0 P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}}.$$

By the Cauchy integral formula, we get the term III_0 . Similar to the discussions in Lemma 3.2 in [PZ], we can get III_a and III_b . \square

Nextly, we compute $P^N \mathcal{O}_2^0 P^N$. We note that it is zero in the Kähler case. Let

$$\begin{aligned} \mathcal{O}'_2 &= \frac{1}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_i) \mathcal{R}, e_s \rangle \nabla_{0,e_i} \nabla_{0,e_s} + \left[\frac{2}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_s) e_s, e_i \rangle \right. \\ &\quad \left. - \left(\frac{1}{2} \sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} + R_{x_0}^E \right) (\mathcal{R}, e_i) \right] \nabla_{0,e_i} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}\nabla_{e_s} \left(\sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0}(\mathcal{R}, e_s) \frac{Z^\alpha}{\alpha!} \right) \\
& -\frac{1}{9} \sum_i \left[\sum_s (\partial_s R^L)_{x_0}(\mathcal{R}, e_i) Z_s \right]^2 \\
& +\frac{1}{12} \left[\sum_s (\nabla_{0,e_s})^2, \langle R_{x_0}^{TX}(\mathcal{R}, e_l) \mathcal{R}, e_l \rangle \right].
\end{aligned} \tag{4.12}$$

Then by (2.13)

$$\begin{aligned}
\mathcal{O}_2^0 &= \mathcal{O}_2' - R_{x_0}^{\wedge^{0,\bullet},B}(\mathcal{R}, e_i) \nabla_{0,e_i} + \left[-(R^E + \frac{1}{2}R^{\det})(w_s, \bar{w}_s) + \frac{r^X}{4} - \frac{1}{4}c(dT_{as}) - \frac{1}{8}|T_{as}|^2 \right. \\
& \left. + 2(R^E + \frac{1}{2}R^{\det})(w_l, \bar{w}_m) \bar{w}^m \wedge i\bar{w}_l \right] (x_0) - \frac{\pi}{2} \sqrt{-1} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} e_i, e_l \rangle c(e_i) c(e_l).
\end{aligned} \tag{4.13}$$

Lemma 4.2 (Lemma 3.6.3 in [LuW2]) *For the operator \mathcal{O}_2' , we have*

$$\begin{aligned}
\mathcal{O}_2' \mathcal{P} &= \left\{ \frac{1}{3} b_i b_s \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_s} \rangle + \frac{1}{2} b_i \sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0}(\mathcal{R}, \partial_{\bar{z}_i}) \frac{Z^\alpha}{\alpha!} \right. \\
& + \frac{4}{3} b_s [\langle R_{x_0}^{TX}(\partial_{z_i}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_s} \rangle - \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{z_i}) \partial_{\bar{z}_i}, \partial_{\bar{z}_s} \rangle] + R^E(\mathcal{R}, \partial_{\bar{z}_s}) b_s \\
& + \langle (\nabla^X \nabla^X \tilde{J})_{(\mathcal{R}, \mathcal{R})} \partial_{z_s}, \partial_{\bar{z}_s} \rangle_{x_0} + 4 \langle R_{x_0}^{TX}(\partial_{z_i}, \partial_{z_s}) \partial_{\bar{z}_i}, \partial_{\bar{z}_s} \rangle \\
& \left. - \frac{1}{3} [\mathcal{L}_0, \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{z_s}) \mathcal{R}, \partial_{\bar{z}_s} \rangle] + \frac{1}{9} |(\nabla_{\mathcal{R}} \tilde{J}) \mathcal{R}|^2 \right\} \mathcal{P}.
\end{aligned} \tag{4.14}$$

By (2.25), we have

$$P^N \mathcal{O}_2^0 P^N = I_{\det(\bar{W}^*) \otimes E} \mathcal{P} \mathcal{O}_2^0 \mathcal{P} I_{\det(\bar{W}^*) \otimes E}, \tag{4.15}$$

so we only need compute $\mathcal{P} \mathcal{O}_2^0 \mathcal{P}$. By [MM12, (4.3)], we have for $\phi \in T^*X$, then

$$\phi(e_i) \nabla_{0,e_i} = \phi(\partial_{z_j}) b_j^+ - \phi(\partial_{\bar{z}_j}) b_j. \tag{4.16}$$

Recall

$$b_l^+ \mathcal{P} = 0 \quad \text{and} \quad (b_l \mathcal{P})(Z, Z') = 2\pi(\bar{z}_l - \bar{z}_l') \mathcal{P}(Z, Z'). \tag{4.17}$$

By Theorem 2.5 and (2.23) and (4.17), we have

$$\mathcal{P} b^\beta z_\alpha \mathcal{P} = 0, \quad \mathcal{P} b^\beta \bar{z}_\alpha \mathcal{P} = 0. \tag{4.18}$$

By (4.16), (4.17) and (4.18), we get

$$\mathcal{P} R_{x_0}^{\wedge^{0,\bullet},B}(\mathcal{R}, e_i) \nabla_{0,e_i} \mathcal{P} = -2R_{x_0}^{\wedge^{0,\bullet},B}(\partial_{z_s}, \partial_{\bar{z}_s}) \mathcal{P}. \tag{4.19}$$

By $\mathcal{P}^2 = \mathcal{P}$, then

$$\begin{aligned} & \mathcal{P} \left[- (R^E + \frac{1}{2} R^{\det})(w_s, \bar{w}_s) + \frac{r^X}{4} - \frac{1}{8} |T_{as}|^2 \right] (x_0) \mathcal{P} \\ &= \left[- (R^E + \frac{1}{2} R^{\det})(w_s, \bar{w}_s) + \frac{r^X}{4} - \frac{1}{8} |T_{as}|^2 \right] (x_0) \mathcal{P}. \end{aligned} \quad (4.20)$$

By (2.11) and direct computations, then

$$\begin{aligned} & I_{\det(\bar{W}^*) \otimes E} \mathcal{P} \left[2(R^E + \frac{1}{2} R^{\det})(w_l, \bar{w}_m) \bar{w}^m \wedge i \bar{w}_l \right] (x_0) \mathcal{P} I_{\det(\bar{W}^*) \otimes E} \\ &= 2 \sum_{m=1}^q (R^E + \frac{1}{2} R^{\det})(w_m, \bar{w}_m) (x_0) \mathcal{P} I_{\det(\bar{W}^*) \otimes E}; \end{aligned} \quad (4.21)$$

$$\begin{aligned} & I_{\det(\bar{W}^*) \otimes E} \mathcal{P} \left[-\frac{1}{4} c(dT_{as}) \right] (x_0) \mathcal{P} I_{\det(\bar{W}^*) \otimes E} = \left[-\frac{1}{8} dT_{as}(w_i, \bar{w}_i, w_s, \bar{w}_s) \right. \\ & \left. + \sum_{i=1}^q dT_{as}(w_i, \bar{w}_i, w_s, \bar{w}_s) - 2 \sum_{i,l=1}^q dT_{as}(w_i, w_l, \bar{w}_i, \bar{w}_l) \right] \mathcal{P} I_{\det(\bar{W}^*) \otimes E}; \end{aligned} \quad (4.22)$$

Similar to (4.19), we have

$$\mathcal{P} R_{x_0}^E(\mathcal{R}, \partial_{\bar{z}_i}) b_i \mathcal{P} = 2 R_{x_0}^E(\partial_{z_i}, \partial_{\bar{z}_i}) \mathcal{P}. \quad (4.23)$$

By (2.22), (2.23) and (2.24), we have

$$(\mathcal{P} z^\alpha \bar{z}^\beta \mathcal{P})(0, 0) = \frac{\alpha!}{\pi^{|\alpha|}} \delta_{\alpha\beta}. \quad (4.24)$$

By (4.24), then

$$\begin{aligned} & \mathcal{P} \left[\left\langle (\nabla^X \nabla^X \tilde{J})_{(\mathcal{R}, \mathcal{R})} \partial_{z_i}, \partial_{\bar{z}_i} \right\rangle_{x_0} \right] \mathcal{P}(0, 0) \\ &= \frac{1}{4\pi} \left[\left\langle (\nabla^X \nabla^X \tilde{J})_{(u_s, \bar{u}_s)} u_i, \bar{u}_i \right\rangle + \left\langle (\nabla^X \nabla^X \tilde{J})_{(\bar{u}_s, u_s)} u_i, \bar{u}_i \right\rangle \right]. \end{aligned} \quad (4.25)$$

Similar to (4.20), then

$$4 \left(\mathcal{P} \left\langle R_{x_0}^{TX}(\partial_{z_i}, \partial_{z_s}) \partial_{\bar{z}_i}, \partial_{\bar{z}_s} \right\rangle \mathcal{P} \right) (0, 0) = \left\langle R_{x_0}^{TX}(u_i, u_s) \bar{u}_i, \bar{u}_s \right\rangle. \quad (4.26)$$

By the relation $[AB, C] = [A, C]B + A[B, C]$ and (4.18), it holds that

$$-\frac{1}{3} \left(\mathcal{P} \left[\mathcal{L}_0, \left\langle R_{x_0}^{TX}(\mathcal{R}, \partial_{z_s}) \mathcal{R}, \partial_{\bar{z}_s} \right\rangle \right] \mathcal{P} \right) (0, 0) = 0. \quad (4.27)$$

By (4.24),(1.32) and (1.35) and $\Psi(X_0, Y, U, V) = \Psi(U, V, X_0, Y)$, we have

$$\begin{aligned}
& \frac{1}{9} \left(\mathcal{P} |(\nabla_{\mathcal{R}} \tilde{J}) \mathcal{R}|^2 \mathcal{P} \right) (0, 0) \\
&= -\frac{1}{9} \left(\mathcal{P} \left\langle (\nabla_{\mathcal{R}} \tilde{J}) \mathcal{R}, (\nabla_{\mathcal{R}} \tilde{J}) \mathcal{R} \right\rangle \mathcal{P} \right) (0, 0) \\
&= \frac{1}{8} \sum_{s \neq l=1}^n (\mathcal{P} z_s z_l \bar{z}_s \bar{z}_l \mathcal{P}) (0, 0) [\Psi_{sl\bar{s}l} + \Psi_{sl\bar{l}s} + \Psi_{ls\bar{s}l} + \Psi_{ls\bar{l}s} + \Psi_{\bar{l}l\bar{s}s} + \Psi_{\bar{l}l\bar{s}s} \\
&\quad + \Psi_{\bar{l}s\bar{s}l} + \Psi_{\bar{l}s\bar{l}s} + \Psi_{s\bar{l}sl} + \Psi_{s\bar{l}sl} + \Psi_{\bar{l}s\bar{s}l} + \Psi_{\bar{l}s\bar{l}s}] \\
&\quad + \frac{1}{4} \sum_{s=1}^n (\mathcal{P} z_s^2 \bar{z}_s^2 \mathcal{P}) (0, 0) [2\Psi_{ss\bar{s}s} + 2\Psi_{s\bar{s}s\bar{s}} + \Psi_{s\bar{s}s\bar{s}} + \Psi_{\bar{s}s\bar{s}s}] \\
&= \frac{1}{4\pi^2} (2\Psi_{sl\bar{s}l} + 2\Psi_{sl\bar{l}s} + 2\Psi_{s\bar{l}sl} + 2\Psi_{\bar{l}s\bar{s}l} + \Psi_{\bar{l}l\bar{s}s} + \Psi_{\bar{l}l\bar{s}s} + \Psi_{s\bar{l}sl} + \Psi_{\bar{l}s\bar{l}s}), \quad (4.28)
\end{aligned}$$

By (4.18), it holds that

$$\frac{4}{3} \mathcal{P} b_s [\langle R_{x_0}^{TX}(\partial_{z_i}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_s} \rangle - \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{z_i}) \partial_{\bar{z}_i}, \partial_{\bar{z}_s} \rangle] \mathcal{P} = 0, \quad (4.29)$$

Similar to (4.18), we have

$$\mathcal{P} b_l z_s z_{j'} \bar{z}_{j''} \mathcal{P} = \mathcal{P} b_l z_s \bar{z}_{j'} \bar{z}_{j''} \mathcal{P} = 0. \quad (4.30)$$

So by (4.30) and R^L is $(1, 1)$ -form and Z^α for $|\alpha| = 2$ being the combinatorics of $z_s z_{j'}$, $z_s \bar{z}_{j'}$, $\bar{z}_s \bar{z}_{j'}$, then

$$\frac{1}{2} \mathcal{P} b_i \sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0}(\mathcal{R}, \partial_{\bar{z}_i}) \frac{Z^\alpha}{\alpha!} \mathcal{P} = 0. \quad (4.31)$$

Similar to (4.31), we can get

$$\frac{1}{3} \mathcal{P} b_i b_s \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_s} \rangle \mathcal{P} = 0. \quad (4.32)$$

By (2.3), similar to (4.21) and (4.25), we get

$$\begin{aligned}
& -\frac{\pi}{2} \sqrt{-1} I_{\det(\bar{W}^*) \otimes E} [\mathcal{P} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} e_i, e_l \rangle c(e_i) c(e_l) \mathcal{P}] (0, 0) I_{\det(\bar{W}^*) \otimes E} \\
&= -\sqrt{-1} \left[-\frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(u_i, \bar{u}_i)} w_s, \bar{w}_s \rangle - \frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\bar{u}_i, u_i)} w_s, \bar{w}_s \rangle \right. \\
&\quad \left. + \sum_{m=1}^q \langle (\nabla^B \nabla^B \mathbf{J})_{(u_i, \bar{u}_i)} w_m, \bar{w}_m \rangle + \sum_{m=1}^q \langle (\nabla^B \nabla^B \mathbf{J})_{(\bar{u}_i, u_i)} w_m, \bar{w}_m \rangle \right] I_{\det(\bar{W}^*) \otimes E}. \quad (4.33)
\end{aligned}$$

By Theorem 2.4, (4.12)-(4.15), (4.19)-(4.23), (4.25)-(4.27), (4.31)-(4.33), we get

Lemma 4.3 *The following equation holds*

$$\begin{aligned}
& (P^N \mathcal{O}_2^0 P^N)(0, 0) \\
= & I_{\det(\overline{W}^*) \otimes E} \left\{ R^{\wedge^{0,\bullet}, B}(u_s, \overline{u}_s) - (R^E + \frac{1}{2} R^{\det})(w_s, \overline{w}_s) + \frac{r^X}{4} - \frac{1}{8} |T_{as}|^2 \right. \\
& + 2 \sum_{m=1}^q (R^E + \frac{1}{2} R^{\det})(w_m, \overline{w}_m) - \frac{1}{8} dT_{as}(w_i, \overline{w}_i, w_s, \overline{w}_s) \\
& + \sum_{i=1}^q dT_{as}(w_i, \overline{w}_i, w_s, \overline{w}_s) - 2 \sum_{i,l=1}^q dT_{as}(w_i, w_l, \overline{w}_i, \overline{w}_l) + R^E(u_s, \overline{u}_s) \\
& + \frac{1}{4\pi} \left[\left\langle (\nabla^X \nabla^X \tilde{J})_{(u_s, \overline{u}_s)} u_i, \overline{u}_i \right\rangle + \left\langle (\nabla^X \nabla^X \tilde{J})_{(\overline{u}_s, u_s)} u_i, \overline{u}_i \right\rangle \right] \\
& + \left\langle R_{x_0}^{TX}(u_i, u_s) \overline{u}_i, \overline{u}_s \right\rangle + \frac{1}{4\pi^2} (2\Psi_{st\bar{s}\bar{l}} + 2\Psi_{sl\bar{l}\bar{s}} + 2\Psi_{s\bar{l}\bar{s}l}) \\
& + 2\Psi_{\bar{l}l\bar{s}\bar{s}} + \Psi_{\bar{l}l\bar{s}\bar{s}} + \Psi_{\bar{l}l\bar{s}\bar{s}} + \Psi_{s\bar{l}\bar{l}\bar{s}} + \Psi_{\bar{l}s\bar{s}l}) \\
& - \sqrt{-1} \left[-\frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(u_i, \overline{u}_i)} w_s, \overline{w}_s \rangle - \frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\overline{u}_i, u_i)} w_s, \overline{w}_s \rangle \right. \\
& \left. + \sum_{m=1}^q \langle (\nabla^B \nabla^B \mathbf{J})_{(u_i, \overline{u}_i)} w_m, \overline{w}_m \rangle + \sum_{m=1}^q \langle (\nabla^B \nabla^B \mathbf{J})_{(\overline{u}_i, u_i)} w_m, \overline{w}_m \rangle \right] \Big\} (x_0) I_{\det(\overline{W}^*) \otimes E}.
\end{aligned} \tag{4.34}$$

By the definition of \mathcal{L}_0 , we have

$$\begin{aligned}
& (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{l-1} (\mathcal{L}_0^{-2} \mathcal{O}_2^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-l} \mathcal{P} \\
= & \frac{1}{(4\pi)^{j-\frac{q}{2}+1} 2^{j-\frac{q}{2}+1} (j-\frac{q}{2})! (j-\frac{q}{2}-l+1)!} (\mathcal{R}_{x_0}^\perp)^{j-\frac{q}{2}} \mathcal{P},
\end{aligned} \tag{4.35}$$

and

$$\mathcal{P}(\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}} I_{2j} = [(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} \mathcal{P}]^* = \frac{1}{(4\pi)^{j-\frac{q}{2}} 2^{j-\frac{q}{2}} (j-\frac{q}{2})!} \mathcal{P}(\mathcal{R}_{x_0}^{\perp,*})^{j-\frac{q}{2}}. \tag{4.36}$$

By (4.9), (4.35) and (4.36), we have

$$\text{III}_0(0, 0) = \left(\sum_{l=1}^{j-\frac{q}{2}} \frac{1}{l} \right) \frac{1}{(4\pi)^{2j-q+1} 2^{2j-q} ((j-\frac{q}{2})!)^2} (\mathcal{R}_{x_0}^\perp)^{j-\frac{q}{2}} (P^N \mathcal{O}_2^0 P^N)(0, 0) (\mathcal{R}_{x_0}^{\perp,*})^{j-\frac{q}{2}} I_{2j}. \tag{4.37}$$

By (4.34) and (4.37), we get $\text{III}_0(0, 0)$.

Nextly **we compute** III_a . We know that $\wedge^{(0,*)}(TX) = \wedge^*(\overline{W}^*) \otimes \wedge^*(\overline{W}^{\perp,*})$ has double degree. Let $\tilde{R}^{\wedge^{0,\bullet}, B}$ and $\frac{1}{4} \widetilde{dT_{as}}$ be the preserving double degree parts of $R^{\wedge^{0,\bullet}, B}$ and $\frac{1}{4} dT_{as}$ respectively. Then

$$\mathcal{O}_2^0 = \mathcal{O}_2^{0,1} + \mathcal{O}_2^{0,2}, \tag{4.38}$$

where

$$\begin{aligned}
\mathcal{O}_2^{0,1} = & \frac{1}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_i) \mathcal{R}, e_s \rangle \nabla_{0,e_i} \nabla_{0,e_s} + \left[\frac{2}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_s) e_s, e_i \rangle \right. \\
& - \left(\frac{1}{2} \sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} + R_{x_0}^E + \tilde{R}_{x_0}^{\wedge^{0,\bullet},B} \right) (\mathcal{R}, e_i) \left. \right] \nabla_{0,e_i} \\
& - \frac{1}{4} \nabla_{e_s} \left(\sum_{|\alpha|=2} (\partial_\alpha R^L)_{x_0}(\mathcal{R}, e_s) \frac{Z^\alpha}{\alpha!} \right) \\
& - \frac{1}{9} \sum_i \left[\sum_s (\partial_s R^L)_{x_0}(\mathcal{R}, e_i) Z_s \right]^2 \\
& + \left[-(R^E + \frac{1}{2} R^{\det})(w_l, \bar{w}_l) + \frac{r^X}{4} - \frac{1}{4} c(\widetilde{dT_{as}}) \right. \\
& - \frac{1}{8} |T_{as}|^2 + 2 \left(\sum_{l,m \leq q} + \sum_{l,m \geq q+1} \right) (R^E + \frac{1}{2} R^{\det})(w_l, \bar{w}_m) \bar{w}^m \wedge i_{\bar{w}_l} \left. \right] \\
& + \frac{1}{12} \left[\sum_s (\nabla_{0,e_s})^2, \langle R_{x_0}^{TX}(\mathcal{R}, e_l) \mathcal{R}, e_l \rangle \right] \\
& - \frac{\pi}{2} \sqrt{-1} \left[-\frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_l \rangle \right. \\
& \left. + \left(\sum_{l,m \leq q} + \sum_{l,m \geq q+1} \right) \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l} \right]; \tag{4.39}
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_2^{0,2} = & \left(\tilde{R}_{x_0}^{\wedge^{0,\bullet},B} - R_{x_0}^{\wedge^{0,\bullet},B} \right) (\mathcal{R}, e_i) \nabla_{0,e_i} + \frac{1}{4} (c(\widetilde{dT_{as}}) - c(dT_{as})) \\
& + 2 \left(\sum_{l \leq q, m \geq q+1} + \sum_{l \geq q+1, m \leq q} \right) (R^E + \frac{1}{2} R^{\det})(w_l, \bar{w}_m) \bar{w}^m \wedge i_{\bar{w}_l} \\
& - \frac{\pi}{2} \sqrt{-1} \left(\sum_{l \leq q, m \geq q+1} + \sum_{l \geq q+1, m \leq q} \right) \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l}. \tag{4.40}
\end{aligned}$$

Then $\mathcal{O}_2^{0,1}$ is the preserving double degree part of \mathcal{O}_2^0 and $\mathcal{O}_2^{0,2}$ is the changing double degree part of \mathcal{O}_2^0 .

By (4.12), (4.39) and (4.16), we have

$$\begin{aligned}
& I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} (\mathcal{L}_0^{-1} \mathcal{O}_2^{0,1}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^k P^N \\
& = \frac{1}{(4\pi)^k 2^k k!} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} (\mathcal{L}_0^{-1} \mathcal{O}_2^0) (\mathcal{R}^\perp)_{x_0}^k \mathcal{P} \\
& = \frac{1}{(4\pi)^k 2^k k!} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} \left\{ R^E(\mathcal{R}, \partial_{\bar{z}_s}) b_s + \tilde{R}_{x_0}^{\wedge^{0,\bullet},B}(\mathcal{R}, \partial_{\bar{z}_s}) b_s \right.
\end{aligned}$$

$$\begin{aligned}
& -(R^E + \frac{1}{2}R^{\det})(w_s, \bar{w}_s) + \frac{r^X}{4} - \frac{1}{4}c(\widetilde{dT_{as}}) - \frac{1}{8}|T_{as}|^2 \\
& + 2(\sum_{l,m \leq q} + \sum_{l,m \geq q+1})(R^E + \frac{1}{2}R^{\det})(w_l, \bar{w}_m)\bar{w}^m \wedge i\bar{w}_l \\
& - \frac{\pi}{2}\sqrt{-1} \left[-\frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_l \rangle \right. \\
& \left. + (\sum_{l,m \leq q} + \sum_{l,m \geq q+1}) \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i\bar{w}_l \right] \Big\} (x_0)(\mathcal{R}^\perp)_{x_0}^k P^N \\
& + \frac{1}{(4\pi)^k 2^k k!} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k [\mathcal{O}'_2 + R_{x_0}^E(\mathcal{R}, e_s) \nabla_{0, e_s}] P^N, \quad (4.41)
\end{aligned}$$

By (2.20) and (2.21) and (4.17), we have

$$\begin{aligned}
& (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\mathcal{R}, \partial_{\bar{z}_s}) b_s (\mathcal{R}^\perp)_{x_0}^k \mathcal{P} = (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} \\
& \cdot \left[(R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{z_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k b_s z_l \mathcal{P} + 2(R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{z_s}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k \mathcal{P} \right. \\
& \left. + \frac{1}{2\pi} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{\bar{z}_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k b_s b_l \mathcal{P} + (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{\bar{z}_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k b_s \bar{z}_l' \mathcal{P} \right]. \quad (4.42)
\end{aligned}$$

By (4.17), direct computations show that

$$\begin{aligned}
& \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{z_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k b_s z_l \mathcal{P} \right\} (0, Z) \\
& = \frac{-2}{(4\pi)^{j-\frac{q}{2}-k+1}} B_1^{k, j-\frac{q}{2}} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}-k} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{z_s}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k \mathcal{P}(0, Z); \quad (4.43)
\end{aligned}$$

$$\begin{aligned}
& 2 \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{z_s}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k \mathcal{P} \right\} (0, Z) \\
& = \frac{2}{(4\pi)^{j-\frac{q}{2}-k+1}} B_0^{k, j-\frac{q}{2}} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}-k} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{z_s}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k \mathcal{P}(0, Z); \quad (4.44)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2\pi} \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{\bar{z}_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k b_s b_l \mathcal{P} \right\} (0, Z) \\
& = \frac{2\pi}{(4\pi)^{j-\frac{q}{2}-k+1}} B_2^{k, j-\frac{q}{2}} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}-k} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{\bar{z}_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k \bar{z}_s \bar{z}_l (\mathcal{P}(0, Z)); \quad (4.45)
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{\bar{z}_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k b_s \bar{z}_l' \mathcal{P} \right\} (0, Z) \\
& = \frac{-2\pi}{(4\pi)^{j-\frac{q}{2}-k+1}} B_1^{k, j-\frac{q}{2}} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}-k} (R^E + \tilde{R}^{\wedge^{0,\bullet}, B})(\partial_{\bar{z}_l}, \partial_{\bar{z}_s})(\mathcal{R}^\perp)_{x_0}^k \bar{z}_s \bar{z}_l (\mathcal{P}(0, Z)). \quad (4.46)
\end{aligned}$$

Similarly

$$\begin{aligned}
& I_{2j}(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}-k}\mathcal{L}_0^{-1}\left[-(R^E+\frac{1}{2}R^{\det})(w_s,\overline{w}_s)+\frac{r^X}{4}-\frac{1}{4}\widetilde{c(dT_{as})}\right. \\
& \left.-\frac{1}{8}|T_{as}|^2+2(\sum_{l,m\leq q}+\sum_{l,m\geq q+1})(R^E+\frac{1}{2}R^{\det})(w_l,\overline{w}_m)\overline{w}^m\wedge i\overline{w}_l\right](x_0)(\mathcal{R}^\perp)_{x_0}^k\mathcal{P} \\
& =\frac{1}{(4\pi)^{j-\frac{q}{2}-k+1}}B_0^{k,j-\frac{q}{2}}(\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}-k}\left[-(R^E+\frac{1}{2}R^{\det})(w_s,\overline{w}_s)+\frac{r^X}{4}-\frac{1}{4}\widetilde{c(dT_{as})}\right. \\
& \left.-\frac{1}{8}|T_{as}|^2+2(\sum_{l,m\leq q}+\sum_{l,m\geq q+1})(R^E+\frac{1}{2}R^{\det})(w_l,\overline{w}_m)\overline{w}^m\wedge i\overline{w}_l\right](x_0)(\mathcal{R}^\perp)_{x_0}^k\mathcal{P}. \quad (4.47)
\end{aligned}$$

Recall [MM08, (2.7)], by (1.33) and R^L is a $(1,1)$ -form, we have

$$b_i\sum_{|\alpha|=2}(\partial_\alpha R^L)_{x_0}(\mathcal{R},\partial_{\overline{z}_i})\frac{Z^\alpha}{\alpha!}=b_iz_l\widehat{\Psi}(\mathcal{R},\mathcal{R},\partial_{z_l},\partial_{\overline{z}_i}). \quad (4.48)$$

By (2.20) and (4.24), we have

$$(b_lz_\alpha z_\beta z_\gamma \mathcal{P})(0,Z)=0; \quad (b_lb_\gamma z_\alpha z_\beta \mathcal{P})(0,Z)=4(\delta_{l\alpha}\delta_{\gamma\beta}+\delta_{l\beta}\delta_{\gamma\alpha})\mathcal{P}(0,Z); \quad (4.49)$$

$$(b_lz_\alpha z_\beta \overline{z}_\gamma \mathcal{P})(0,Z)=\frac{1}{2\pi}[(b_lb_\gamma z_\alpha z_\beta+2\delta_{\gamma\beta}b_jz_\alpha+2\delta_{\gamma\alpha}b_lz_\beta)\mathcal{P}](0,Z); \quad (4.50)$$

$$\int_{\mathbb{R}^{2n}}(b_lz_\alpha \overline{z}_\beta \overline{z}_\gamma \mathcal{P})(0,Z)\mathcal{P}(Z,0)d\nu_{TX}(Z)=0; \quad (4.51)$$

$$\int_{\mathbb{R}^{2n}}(b_l\overline{z}_\alpha \overline{z}_\beta \overline{z}_\gamma \mathcal{P})(0,Z)\mathcal{P}(Z,0)d\nu_{TX}(Z)=0. \quad (4.52)$$

By (4.48)-(4.52), then

$$\begin{aligned}
& \left(b_i\sum_{|\alpha|=2}(\partial_\alpha R^L)_{x_0}(\mathcal{R},\partial_{\overline{z}_i})\frac{Z^\alpha}{\alpha!}\mathcal{P}\right)(0,Z) \\
& =[\widehat{\Psi}(\partial_{z_\beta},\partial_{\overline{z}_\gamma},\partial_{z_\alpha},\partial_{\overline{z}_i})+\widehat{\Psi}(\partial_{\overline{z}_\gamma},\partial_{z_\beta},\partial_{z_\alpha},\partial_{\overline{z}_i})](b_iz_\alpha z_\beta \overline{z}_\gamma \mathcal{P})(0,Z). \quad (4.53)
\end{aligned}$$

By (4.49),(4.50) and (4.53), then

$$\begin{aligned}
& \left\{(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}-k}\mathcal{L}_0^{-1}(\mathcal{R}^\perp)_{x_0}^k\left[\frac{1}{2}b_i\sum_{|\alpha|=2}(\partial_\alpha R^L)_{x_0}(\mathcal{R},\partial_{\overline{z}_i})\frac{Z^\alpha}{\alpha!}\right]\mathcal{P}\right\}(0,Z) \\
& =\frac{1}{(4\pi)^{j-\frac{q}{2}-k+2}}(B_2^{k,j-\frac{q}{2}}-B_1^{k,j-\frac{q}{2}})(\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}}(\widehat{\Psi}_{\beta\overline{\beta}\alpha\overline{\alpha}}+\widehat{\Psi}_{\overline{\alpha}\beta\alpha\overline{\beta}}). \quad (4.54)
\end{aligned}$$

It holds that

$$b_i b_l \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_l} \rangle = \frac{1}{4} (R_{\alpha\bar{i}\beta\bar{l}}^X b_i b_l z_\alpha z_\beta + R_{\alpha\bar{i}\beta\bar{l}}^X b_i b_l \bar{z}_\alpha \bar{z}_\beta + 2R_{\alpha\bar{i}\beta\bar{l}}^X b_i b_l \bar{z}_\alpha z_\beta). \quad (4.55)$$

By (4.49), then

$$\begin{aligned} & \frac{1}{12} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k R_{\alpha\bar{i}\beta\bar{l}}^X b_i b_l z_\alpha z_\beta \mathcal{P} \right] (0, Z) \\ &= \frac{1}{3} \frac{1}{(4\pi)^{j-\frac{q}{2}-k+1}} B_2^{j-\frac{q}{2}, k} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} (R_{\alpha\bar{\alpha}\beta\bar{\beta}}^X + R_{\alpha\bar{\beta}\beta\bar{\alpha}}^X) \mathcal{P}(0, Z). \end{aligned} \quad (4.56)$$

By (4.17), then

$$(b_i b_l \bar{z}_\alpha \bar{z}_\beta \mathcal{P})(0, Z) = \left[\left(\frac{b_i b_l b_\alpha b_\beta}{4\pi^2} + \frac{1}{2\pi} b_i b_l b_\beta \bar{z}'_\alpha + \frac{1}{2\pi} b_i b_l b_\alpha \bar{z}'_\beta + b_i b_l \bar{z}'_\alpha \bar{z}'_\beta \right) \mathcal{P} \right] (0, Z). \quad (4.57)$$

By (4.18) and (4.47), we have

$$\int_{\mathbb{R}^{2n}} (b_i b_l \bar{z}_\alpha \bar{z}_\beta \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0; \quad (4.58)$$

$$\int_{\mathbb{R}^{2n}} \frac{1}{12} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k R_{\alpha\bar{i}\beta\bar{l}}^X b_i b_l \bar{z}_\alpha \bar{z}_\beta \mathcal{P} \right] (0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0; \quad (4.59)$$

Similar to (4.59), by (2.20) and (4.24), we have

$$(b_i b_l \bar{z}_\alpha z_\beta \mathcal{P})(0, Z) = \left[\frac{b_i b_l b_\alpha z_\beta}{2\pi} + \frac{\delta_{\alpha\beta} b_i b_l}{\pi} + b_i b_l z_\beta \bar{z}'_\alpha \right] \mathcal{P}(0, Z), \quad (4.60)$$

$$\int_{\mathbb{R}^{2n}} \frac{1}{6} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k R_{\alpha\bar{i}\beta\bar{l}}^X b_i b_l \bar{z}_\alpha z_\beta \mathcal{P} \right] (0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0. \quad (4.61)$$

By (4.55), (4.56), (4.59) and (4.61), then

$$\begin{aligned} & \frac{1}{3} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k b_i b_l \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_l} \rangle \mathcal{P} \right] (0, Z) \\ &= \frac{1}{3} \frac{1}{(4\pi)^{j-\frac{q}{2}-k+1}} B_2^{j-\frac{q}{2}, k} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} (R_{\alpha\bar{\alpha}\beta\bar{\beta}}^X + R_{\alpha\bar{\beta}\beta\bar{\alpha}}^X) \mathcal{P}(0, Z). \end{aligned} \quad (4.62)$$

We see that

$$\begin{aligned} & \frac{4}{3} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k b_r \left[\langle R_{x_0}^{TX}(\partial_{z_i}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_r} \rangle - \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{z_i}) \partial_{\bar{z}_i}, \partial_{\bar{z}_r} \rangle \right] \mathcal{P} \right] (0, Z) \\ &= \frac{1}{3} \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k \left[(R_{i\bar{i}l\bar{r}}^X - R_{li\bar{i}\bar{r}}^X) b_r z_l + (R_{i\bar{i}l\bar{r}}^X - R_{li\bar{i}\bar{r}}^X) b_r \bar{z}_l \right] \mathcal{P} \right\} (0, Z), \\ & \frac{1}{3} \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k \left[(R_{i\bar{i}l\bar{r}}^X - R_{li\bar{i}\bar{r}}^X) b_r z_l \right] \mathcal{P} \right\} (0, Z) \end{aligned} \quad (4.63)$$

$$= \frac{-2}{3 \times (4\pi)^{j-\frac{q}{2}-k+1}} B_1^{k,j-\frac{q}{2}} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} (R_{i\bar{i}l\bar{l}}^X - R_{li\bar{l}\bar{i}}^X) \mathcal{P}(0, Z), \quad (4.64)$$

$$\int_{\mathbb{R}^{2n}} \frac{1}{3} \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k (R_{i\bar{i}l\bar{l}}^X - R_{li\bar{l}\bar{i}}^X) b_r \bar{z}_l \mathcal{P} \right\} (0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0. \quad (4.65)$$

By (4.63)-(4.65), then

$$\begin{aligned} & \frac{4}{3} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k b_r \left[\langle R_{x_0}^{TX}(\partial_{z_i}, \partial_{\bar{z}_i}) \mathcal{R}, \partial_{\bar{z}_r} \rangle - \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{z_i}) \partial_{\bar{z}_i}, \partial_{\bar{z}_r} \rangle \right] \mathcal{P} \right] (0, Z) \\ &= \frac{-2}{3 \times (4\pi)^{j-\frac{q}{2}-k+1}} B_1^{k,j-\frac{q}{2}} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} (R_{i\bar{i}l\bar{l}}^X - R_{li\bar{l}\bar{i}}^X) \mathcal{P}(0, Z), \end{aligned} \quad (4.66)$$

Similar to (4.27) and (4.65), we have

$$\begin{aligned} & -\frac{1}{3} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k \left[\mathcal{L}_0, \langle R_{x_0}^{TX}(\mathcal{R}, \partial_{z_j}) \mathcal{R}, \partial_{\bar{z}_j} \rangle \right] \mathcal{P} \right] (0, Z) \\ &= -\frac{1}{6} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k (R_{\bar{r}l\alpha\bar{l}}^X + R_{\alpha l\bar{r}\bar{l}}^X) b_r z_\alpha \mathcal{P} \right] (0, Z) \\ & -\frac{1}{6} \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k (R_{\bar{r}l\alpha\bar{l}}^X + R_{\alpha l\bar{r}\bar{l}}^X) b_r \bar{z}_\alpha \mathcal{P} \right] (0, Z) \\ &= \frac{1}{3} \frac{1}{(4\pi)^{j-\frac{q}{2}-k+1}} B_1^{k,j-\frac{q}{2}} (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} (R_{\bar{r}l\alpha\bar{l}}^X + R_{\alpha l\bar{r}\bar{l}}^X) \mathcal{P}(0, Z). \end{aligned} \quad (4.67)$$

By (1.34) and

$$z_l \bar{z}_\alpha \mathcal{P} = \left(\frac{b_\alpha z_l}{2\pi} + \frac{\delta_{\alpha l}}{\pi} + z_l \bar{z}'_\alpha \right) \mathcal{P}, \quad (4.68)$$

then

$$\begin{aligned} & \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k \Phi(\mathcal{R}, \mathcal{R}) \mathcal{P} \right] (0, Z) \\ &= \left\{ (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k \left([\Phi(\partial_{z_l}, \partial_{\bar{z}_\alpha}) + \Phi(\partial_{\bar{z}_\alpha}, \partial_{z_l})] z_l \bar{z}_\alpha + \Phi(\partial_{z_l}, \partial_{\bar{z}_\alpha}) \bar{z}_l \bar{z}_\alpha \right) \mathcal{P} \right\} (0, Z) \\ &= \frac{1}{2\pi} \frac{1}{(4\pi)^{j-\frac{q}{2}-k+1}} (B_0^{k,j-\frac{q}{2}} - B_1^{k,j-\frac{q}{2}}) (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} [\Phi(w_l, \bar{w}_l) + \Phi(\bar{w}_l, w_l)] \mathcal{P}(0, Z). \end{aligned} \quad (4.69)$$

We know that

$$(z_l z_m z_\beta z_\alpha \mathcal{P})(0, Z) = 0, \quad (z_l z_m z_\beta \bar{z}_\alpha \mathcal{P})(0, Z) = \frac{1}{2\pi} (b_\alpha z_l z_m z_\beta \mathcal{P})(0, Z) = 0; \quad (4.70)$$

$$\begin{aligned} (z_l z_m \bar{z}_\beta \bar{z}_\alpha \mathcal{P})(0, Z) &= \frac{1}{4\pi^2} [(b_\beta b_\alpha z_l z_m + 2\delta_{l\alpha} b_\beta z_m + 2\delta_{m\alpha} b_\beta z_l \\ &+ 2\delta_{\beta l} b_\alpha z_m + 2\delta_{m\beta} b_\alpha z_l + 4\delta_{m\beta} \delta_{l\alpha} + 4\delta_{\beta l} \delta_{m\alpha}) \mathcal{P}](0, Z); \end{aligned} \quad (4.71)$$

$$\int_{\mathbb{R}^{2n}} (z_l \bar{z}_m \bar{z}_\beta \bar{z}_\alpha \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0; \quad (4.72)$$

$$\int_{\mathbb{R}^{2n}} (\bar{z}_l \bar{z}_m \bar{z}_\beta \bar{z}_\alpha \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0. \quad (4.73)$$

Let the coefficient of $z_l z_m \bar{z}_\beta \bar{z}_\alpha$ for $l \leq m$ and $\beta \leq \alpha$ in the expansion of $\Psi(\mathcal{R}, \mathcal{R}, \mathcal{R}, \mathcal{R})$ be $A_{lm\bar{\beta}\bar{\alpha}}$, then by (4.70)-(4.73), we have

$$\begin{aligned} & \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k \Psi(\mathcal{R}, \mathcal{R}, \mathcal{R}, \mathcal{R}) \mathcal{P} \right] (0, Z) \\ &= \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} (\mathcal{R}^\perp)_{x_0}^k \sum_{l \leq m} \sum_{\beta \leq \alpha} A_{lm\bar{\beta}\bar{\alpha}} z_l z_m \bar{z}_\beta \bar{z}_\alpha \mathcal{P} \right] (0, Z) \\ &= \frac{1}{\pi^2} \frac{1}{(4\pi)^{j-\frac{q}{2}-k+1}} (B_0^{k,j-\frac{q}{2}} + B_2^{k,j-\frac{q}{2}} - 2B_1^{k,j-\frac{q}{2}}) (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} (2A_{ll\bar{l}\bar{l}} + \sum_{l < m} A_{lm\bar{l}\bar{m}}) \\ &= \frac{1}{4\pi^2} \frac{1}{(4\pi)^{j-\frac{q}{2}-k+1}} (B_0^{k,j-\frac{q}{2}} + B_2^{k,j-\frac{q}{2}} - 2B_1^{k,j-\frac{q}{2}}) (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}} \\ & \quad \times (2\Psi_{sl\bar{s}\bar{l}} + 2\Psi_{sl\bar{l}\bar{s}} + 2\Psi_{s\bar{l}\bar{s}l} + 2\Psi_{l\bar{l}s\bar{s}} + \Psi_{l\bar{l}s\bar{s}} + \Psi_{l\bar{l}s\bar{s}} + \Psi_{s\bar{l}\bar{l}s} + \Psi_{l\bar{s}\bar{s}l}). \quad (4.74) \end{aligned}$$

By (4.24) and direct computations, we have

$$\begin{aligned} & -\frac{1}{(4\pi)^k 2^k k!} \frac{\pi \sqrt{-1}}{2} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} \mathcal{L}_0^{-1} \left[-\frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_l \rangle \right. \\ & \quad \left. + \left(\sum_{l, m \leq q} + \sum_{l, m \geq q+1} \right) \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i\bar{w}_l \right] (x_0) (\mathcal{R}^\perp)_{x_0}^k P^N(0, Z) \\ &= \frac{\sqrt{-1}}{2(4\pi)^{j-\frac{q}{2}+1}} B_0^{1,k} (B_1^{k,j-\frac{q}{2}} - B_0^{k,j-\frac{q}{2}}) (\mathcal{R}^\perp)_{x_0}^{j-\frac{q}{2}-k} \\ & \quad \left[-\frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha})} w_l, \bar{w}_l \rangle - \frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{\bar{z}_\alpha}, \partial_{z_\alpha})} w_l, \bar{w}_l \rangle \right. \\ & \quad \left. + \left(\sum_{l, m \leq q} + \sum_{l, m \geq q+1} \right) \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i\bar{w}_l \right. \\ & \quad \left. + \left(\sum_{l, m \leq q} + \sum_{l, m \geq q+1} \right) \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{\bar{z}_\alpha}, \partial_{z_\alpha})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i\bar{w}_l \right] (\mathcal{R}^\perp)_{x_0}^k P^N(0, Z). \quad (4.75) \end{aligned}$$

By (4.41)-(4.47), (4.54), (4.56), (4.62), (4.66), (4.67), (4.69), (4.74) and (4.75), we get

$$\begin{aligned} & I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} (\mathcal{L}_0^{-1} \mathcal{O}_2^{0,1}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^k P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}} I_{2j} \\ &= \frac{1}{(4\pi)^{2j-q+1} 2^{j-\frac{q}{2}+k} (j-\frac{q}{2})! k!} I_{2j} (\mathcal{R}^\perp)(x)^{j-\frac{q}{2}-k} \\ & \quad \cdot \left\{ -B_1^{k,j-\frac{q}{2}} R^E(w_s, \bar{w}_s) + (B_0^{k,j-\frac{q}{2}} - B_1^{k,j-\frac{q}{2}}) \tilde{R}^{\wedge^{0,\bullet}, B}(w_s, \bar{w}_s) \right\} \end{aligned} \quad (4.76)$$

$$\begin{aligned}
& + B_0^{k,j-\frac{q}{2}} \left[2 \left(\sum_{l,m \leq q} + \sum_{l,m \geq q+1} \right) (R^E + \frac{1}{2} R^{\det})(w_s, \overline{w}_m) \overline{w}^m \wedge i \overline{w}_s \right. \\
& - \frac{1}{4} c(\widetilde{dT_{as}}) + \frac{r^X}{4} - \frac{1}{8} |T_{as}|^2 - \frac{1}{2} R^{\det}(w_s, \overline{w}_s) \left. \right] + \frac{\sqrt{-1}}{2} (B_1^{k,j-\frac{q}{2}} - B_0^{k,j-\frac{q}{2}}) \\
& \cdot \left[-\frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{z_\alpha}, \partial_{\overline{z}_\alpha})} w_l, \overline{w}_l \rangle - \frac{1}{2} \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{\overline{z}_\alpha}, \partial_{z_\alpha})} w_l, \overline{w}_l \rangle \right. \\
& + \left(\sum_{l,m \leq q} + \sum_{l,m \geq q+1} \right) \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{z_\alpha}, \partial_{\overline{z}_\alpha})} w_l, \overline{w}_m \rangle \overline{w}^m \wedge i \overline{w}_l \\
& + \left. \left(\sum_{l,m \leq q} + \sum_{l,m \geq q+1} \right) \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{\overline{z}_\alpha}, \partial_{z_\alpha})} w_l, \overline{w}_m \rangle \overline{w}^m \wedge i \overline{w}_l \right] \Big\} \\
& \cdot (\mathcal{R}^\perp)(x)^k I_{\det(\overline{W}^*) \otimes E}(\mathcal{R}^{\perp,*})(x)^{j-\frac{q}{2}} I_{2j} \\
& , \quad + \frac{1}{(4\pi)^{2j-q+1} 2^{j-\frac{q}{2}+k} (j-\frac{q}{2})! k!} I_{2j} \left\{ (B_0^{k,j-\frac{q}{2}} - \frac{1}{3} B_1^{k,j-\frac{q}{2}}) R_{\alpha\beta\alpha\overline{\beta}}^{TX} \right. \\
& + (\frac{1}{3} B_2^{k,j-\frac{q}{2}} - \frac{2}{3} B_1^{k,j-\frac{q}{2}}) R_{\alpha\overline{\alpha}\beta\overline{\beta}}^{TX} + (-\frac{1}{3} B_2^{k,j-\frac{q}{2}} + \frac{1}{3} B_1^{k,j-\frac{q}{2}}) R_{\overline{\alpha}\beta\alpha\overline{\beta}}^{TX} \\
& + \frac{1}{2\pi} (B_0^{k,j-\frac{q}{2}} - B_1^{k,j-\frac{q}{2}}) [\Phi(w_l, \overline{w}_l) + \Phi(\overline{w}_l, w_l)] \\
& + \frac{1}{4\pi^2} (B_0^{k,j-\frac{q}{2}} + B_2^{k,j-\frac{q}{2}} - 2B_1^{k,j-\frac{q}{2}}) \\
& \cdot (2\Psi_{s\overline{l}\overline{s}l} + 2\Psi_{s\overline{l}\overline{s}s} + 2\Psi_{s\overline{l}l\overline{s}} + 2\Psi_{\overline{l}l\overline{s}s} + \Psi_{\overline{l}\overline{s}s} + \Psi_{\overline{l}l\overline{s}} + \Psi_{s\overline{l}\overline{s}} + \Psi_{\overline{l}s\overline{s}l}) \\
& \cdot (B_2^{k,j-\frac{q}{2}} - B_1^{k,j-\frac{q}{2}}) (\widehat{\Psi}_{\beta\overline{\beta}\alpha\overline{\alpha}} + \widehat{\Psi}_{\overline{\alpha}\beta\alpha\overline{\beta}}) \Big\} (\mathcal{R}^\perp)(x)^{j-\frac{q}{2}} I_{\det(\overline{W}^*) \otimes E}(\mathcal{R}^{\perp,*})(x)^{j-\frac{q}{2}} I_{2j}.
\end{aligned}$$

In the following, we assume that $n = 4$, $q = 2$, $j = 2$. Then by $\mathcal{O}_2^{0,2}$ changing the double degree and preserving the total degree, so

$$\begin{aligned}
& \sum_{k=0}^{j-\frac{q}{2}} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-k} (\mathcal{L}_0^{-1} \mathcal{O}_2^{0,2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^k P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}} I_{2j} \\
& = I_4 (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{0,2}) P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1}) I_4 + I_4 (\mathcal{L}_0^{-1} \mathcal{O}_2^{0,2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1}) I_4 \\
& = I_4 (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{0,2}) P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1}) I_4. \tag{4.77}
\end{aligned}$$

By (4.40), we have

$$\begin{aligned}
\mathcal{O}_2^{0,2} P^N & = \left\{ - \sum_{l \leq q, m \geq q+1} \langle R^B(\mathcal{R}, e_i) w_l, \overline{w}_m \rangle \overline{w}^m \wedge i \overline{w}_l \nabla_{0, e_i} \right. \\
& + 2 \sum_{l \leq q, m \geq q+1} (R^E + \frac{1}{2} R^{\det})(w_l, \overline{w}_m) \overline{w}^m \wedge i \overline{w}_l \\
& \left. - \frac{\pi}{2} \sqrt{-1} \sum_{l \leq q, m \geq q+1} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \overline{w}_m \rangle \overline{w}^m \wedge i \overline{w}_l \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{l \leq q, m \geq q+1} dT_{as}(w_l, \bar{w}_m, w_k, \bar{w}_k) \bar{w}^m \wedge i_{\bar{w}_l} \\
& - \frac{1}{4} \sum_{1 \leq i, j' \leq q} \sum_{k \text{ or } l \geq q+1} dT_{as}(w_i, w_{j'}, \bar{w}_k, \bar{w}_l) \bar{w}^k \wedge \bar{w}^l \wedge i_{\bar{w}_i} i_{\bar{w}_{j'}} \Big\} P^N. \quad (4.78)
\end{aligned}$$

By (2.21), we have

$$\begin{aligned}
& I_4(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) \mathcal{L}_0^{-1} \left[2 \sum_{l \leq q, m \geq q+1} (R^E + \frac{1}{2} R^{\det})(w_l, \bar{w}_m) \bar{w}^m \wedge i_{\bar{w}_l} \right. \\
& \quad \left. + \frac{1}{2} \sum_{l \leq q, m \geq q+1} dT_{as}(w_l, \bar{w}_m, w_k, \bar{w}_k) \bar{w}^m \wedge i_{\bar{w}_l} \right] P^N \\
& = \frac{1}{64\pi^2} \mathcal{R}^0 \left[2 \sum_{l \leq q, m \geq q+1} (R^E + \frac{1}{2} R^{\det})(w_l, \bar{w}_m) \bar{w}^m \wedge i_{\bar{w}_l} \right. \\
& \quad \left. + \frac{1}{2} \sum_{l \leq q, m \geq q+1} dT_{as}(w_l, \bar{w}_m, w_k, \bar{w}_k) \bar{w}^m \wedge i_{\bar{w}_l} \right] P^N. \quad (4.79)
\end{aligned}$$

Similar to (4.75), it holds that

$$\begin{aligned}
& I_4(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) \mathcal{L}_0^{-1} \left[-\frac{\pi}{2} \sqrt{-1} \sum_{l \leq q, m \geq q+1} \langle (\nabla^B \nabla^B \mathbf{J})_{(\mathcal{R}, \mathcal{R})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l} \right] P^N(0, Z) \\
& = \frac{-5\sqrt{-1}}{32 \times 36\pi^2} \mathcal{R}^0 \left[\sum_{l \leq q, m \geq q+1} \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l} \right. \\
& \quad \left. + \langle (\nabla^B \nabla^B \mathbf{J})_{(\partial_{\bar{z}_\alpha}, \partial_{z_\alpha})} w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l} \right] P^N(0, Z). \quad (4.80)
\end{aligned}$$

By (4.16) and (4.17), we get

$$\begin{aligned}
& -I_4(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) \mathcal{L}_0^{-1} \left[\sum_{l \leq q, m \geq q+1} \langle R^B(\mathcal{R}, e_i) w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l} \nabla_{0, e_i} \right] P^N(0, Z) \\
& = \frac{5}{288\pi^2} \mathcal{R}^0 \left[\sum_{l \leq q, m \geq q+1} \langle R^B(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l} \right] P^N(0, Z). \quad (4.81)
\end{aligned}$$

Similarly

$$-\frac{1}{4} I_4(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) \mathcal{L}_0^{-1} \sum_{1 \leq i, j' \leq q} \sum_{k \text{ or } l \geq q+1} dT_{as}(w_i, w_{j'}, \bar{w}_k, \bar{w}_l) \bar{w}^k \wedge \bar{w}^l \wedge i_{\bar{w}_i} i_{\bar{w}_{j'}} P^N(0, Z)$$

$$\begin{aligned}
&= -\frac{1}{512\pi^2} \mathcal{R}^\top \sum_{1 \leq i, j' \leq q} \sum_{k, l \geq q+1} dT_{as}(w_i, w_{j'}, \bar{w}_k, \bar{w}_l) \bar{w}^k \wedge \bar{w}^l \wedge i_{\bar{w}_i} i_{\bar{w}_{j'}} P^N(0, Z) \\
&- \frac{1}{256\pi^2} \mathcal{R}^0 \sum_{1 \leq i, j' \leq q} \left(\sum_{k \leq q, l \geq q+1} + \sum_{l \leq q, k \geq q+1} \right) dT_{as}(w_i, w_{j'}, \bar{w}_k, \bar{w}_l) \bar{w}^k \wedge \bar{w}^l \wedge i_{\bar{w}_i} i_{\bar{w}_{j'}} P^N(0, Z).
\end{aligned} \tag{4.82}$$

When $n = 4$, $q = 2$, $j = 2$, we have

$$\text{III}_a(0, 0) = \frac{1}{4\pi} [(4.79) + (4.80) + (4.81) + (4.82)] I_{\det(\bar{W}^*) \otimes E} (\mathcal{R}^\perp)^* I_4 + \sum_{k=0}^1 (4.76). \tag{4.83}$$

4.3 The computations of the term V

In this section, we will compute the term V. By the discussions after (4.2), for $l = 2$, then $4j \leq 2q + 2k - 4 \leq 4j$, so $k = 2j + 2 - q$. There are $2j - q$ \mathcal{O}_{r_i} equal to \mathcal{O}_2 and 2 equal to \mathcal{O}_1 ; By (3.16) and (3.17), when $l = 2$ and $l_1 = 0$, then $i_0 = j - \frac{q}{2} + 3$, when $l = 2$ and $l_1 = 1$, then $i_0 = j - \frac{q}{2} + 2$, when $l = 2$ and $l_1 = 2$, then $i_0 = j - \frac{q}{2} + 1$, so we only have one $\eta_{i_0} = N$. These three cases correspond to the terms V_a , V_b , V_a^* and

$$\begin{aligned}
V_a &= \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} \\
&\quad \times (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1 - m_2} P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j - \frac{q}{2}} I_{2j},
\end{aligned} \tag{4.84}$$

$$\begin{aligned}
V_b &= \sum_{0 \leq m_1, m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1} P^N \\
&\quad \times (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1 \mathcal{L}_0^{-1} P^{N^\perp}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j - \frac{q}{2} - m_2} I_{2j},
\end{aligned} \tag{4.85}$$

$$V = V_a + V_b + V_a^*. \tag{4.86}$$

By (2.12) and (2.3) and Proposition 2.1 in [LuW1], we have

$$\mathcal{O}_1 = \mathcal{O}'_1 + \mathcal{O}''_1, \tag{4.87}$$

where

$$\mathcal{O}'_1 = -\frac{2}{3} (\partial_s R^L)_{x_0} (\mathcal{R}, e_i) Z_s \nabla_{0, e_i} - \frac{1}{3} (\partial_s R^L)_{x_0} (\mathcal{R}, e_s); \tag{4.88}$$

$$\mathcal{O}''_1 = -4\pi \sqrt{-1} \left(\sum_{l \leq q} \sum_{q+1 \leq m} + \sum_{m \leq q} \sum_{q+1 \leq l} \right) \langle (\nabla_{\mathcal{R}}^B \mathbf{J}) w_l, \bar{w}_m \rangle \bar{w}^m \wedge i_{\bar{w}_l}. \tag{4.89}$$

Firstly, we compute the term

$$\begin{aligned}
V_a^0 &:= \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}'_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} \\
&\quad \times (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}'_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1 - m_2} P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j - \frac{q}{2}} I_{2j},
\end{aligned} \tag{4.90}$$

By

$$(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-m_1-m_2} P^N = \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1-m_2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1-m_2} P^N, \quad (4.91)$$

$$\frac{2}{3}(\partial_l R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_i}) Z_l z_\alpha b_i = \frac{2}{3}(\partial_{z_\beta} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_i}) z_\alpha z_\beta b_i + \frac{2}{3}(\partial_{\bar{z}_\beta} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_i}) z_\alpha \bar{z}_\beta b_i, \quad (4.92)$$

and (2.12), (4.16), (4.17) and R^L being a (1, 1) form, then

$$\begin{aligned} \mathcal{O}'_1(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-m_1-m_2} P^N &= \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1-m_2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} \mathcal{R}^{j-\frac{q}{2}-m_1-m_2} \\ &\times \left[\frac{2}{3}(\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) z_\alpha z_{j'} b_l + \frac{2}{3}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) z_\alpha \bar{z}_r b_l \right. \\ &\left. - \frac{1}{3}(\partial_{j'} R^L)_{x_0}(\partial_{z_\alpha}, e_{j'}) z_\alpha - \frac{1}{3}(\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, e_{j'}) \left(\frac{b_\alpha}{2\pi} + \bar{z}'_\alpha \right) \right] P^N. \end{aligned} \quad (4.93)$$

By

$$z_\alpha z_j b_l = b_l z_\alpha z_j + 2\delta_{l\alpha} z_j + 2\delta_{jl} z_\alpha, \quad (4.94)$$

and

$$z_\alpha \bar{z}_r b_l \mathcal{P} = \left(\frac{b_l b_r z_\alpha}{2\pi} + \frac{\delta_{\alpha r} b_l}{\pi} + \frac{\delta_{\alpha l} b_r}{\pi} + b_l z_\alpha \bar{z}'_r + 2\delta_{\alpha l} \bar{z}'_r \right) \mathcal{P}, \quad (4.95)$$

then

$$\begin{aligned} &(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1} \mathcal{O}'_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-m_1-m_2} P^N \\ &= \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_0^{j-\frac{q}{2}-m_1-m_2, j-\frac{q}{2}-m_1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} \\ &\cdot \left[\frac{4}{3}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_l}) \bar{z}'_r + \frac{4}{3}(\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) z_j \right. \\ &+ \frac{4}{3}(\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_j}) z_\alpha - \frac{1}{3}(\partial_{j'} R^L)_{x_0}(\partial_{z_\alpha}, e_{j'}) z_\alpha \\ &\left. - \frac{1}{3}(\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, e_{j'}) \bar{z}'_\alpha \right] P^N \\ &+ \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2, j-\frac{q}{2}-m_1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} \\ &\cdot \left[\frac{2}{3\pi}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_l}) b_l + \frac{2}{3\pi}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) b_r \right. \\ &+ \frac{2}{3}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) b_l z_\alpha \bar{z}'_\gamma + \frac{2}{3}(\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) b_l z_\alpha z_{j'} \\ &\left. - \frac{1}{3}(\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, e_{j'}) \frac{b_\alpha}{2\pi} \right] P^N \\ &+ \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2, j-m_1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} \\ &\times \frac{1}{3\pi}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) b_l b_r z_\alpha P^N. \end{aligned} \quad (4.96)$$

By (4.16) and (4.17) and R^L being a $(1, 1)$ -form, we get

$$\mathcal{O}'_1 = A_1 + A_2 + A_3 + A_4 + A_5, \quad (4.97)$$

where

$$A_1 = -\frac{2}{3}(\partial_{z_\beta} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, \partial_{z_i})\bar{z}_\alpha z_\beta b_i^+ - \frac{2}{3}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, \partial_{z_i})\bar{z}_\alpha \bar{z}_r b_i^+, \quad (4.98)$$

$$\begin{aligned} A_2 = & \left[\frac{4}{3}(\partial_{z_j} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) + \frac{4}{3}(\partial_{z_\alpha} R^L)_{x_0}(\partial_{z_{j'}} , \partial_{\bar{z}_\alpha}) - \frac{1}{3}(\partial_\alpha R^L)_{x_0}(\partial_{z_{j'}} , e_\alpha) \right] z_{j'} \\ & + \left[\frac{4}{3}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_l}) - \frac{1}{3}(\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_r}, e_{j'}) \right] \bar{z}'_r, \end{aligned} \quad (4.99)$$

$$\begin{aligned} A_3 = & \left[\frac{2}{3\pi}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_\alpha}) + \frac{2}{3\pi}(\partial_{\bar{z}_\alpha} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi}(\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, e_{j'}) \right] b_\alpha \\ & + \frac{2}{3}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) b_l z_\alpha \bar{z}'_\gamma + \frac{2}{3}(\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) b_l z_\alpha z_{j'}, \end{aligned} \quad (4.100)$$

$$A_4 = \frac{1}{3\pi}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) b_l b_r z_\alpha. \quad (4.101)$$

$$\begin{aligned} A_5 = & \frac{2}{3}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l})(b_l z_\alpha + 2\delta_{\alpha l})(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) \\ & - \frac{1}{3}(\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_r}, e_{j'})\left(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r\right). \end{aligned} \quad (4.102)$$

By

$$\int_{\mathbb{R}^{2n}} (b_l \bar{z}'_r \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0; \quad (4.103)$$

$$\int_{\mathbb{R}^{2n}} (b_l z_\alpha \bar{z}'_r \bar{z}'_s \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0; \quad (4.104)$$

$$\int_{\mathbb{R}^{2n}} (b_l z_\alpha \bar{z}'_r z_{j'} \mathcal{P})(Z, 0) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0; \quad (4.105)$$

$$(b_l z_{j'} \mathcal{P})(0, Z) = -2\delta_{j'l} \mathcal{P}(0, Z), \quad (4.106)$$

$$(b_l z_\alpha z_{j'} z_s \mathcal{P})(0, Z) = 0 \quad (b_l z_\alpha z_{j'} \bar{z}'_s \mathcal{P})(0, Z) = 0, \quad (4.107)$$

then

$$A_1 A_2 \mathcal{P} = 0; \quad (4.108)$$

$$\int_{\mathbb{R}^{2n}} (A_2^2 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0; \quad (4.109)$$

$$\begin{aligned} & \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1, j-\frac{q}{2}-m_1-m_2} B_0^{j-\frac{q}{2}-m_1-m_2, j-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\ & \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_3 A_2 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\ = & \frac{-2}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1, j-\frac{q}{2}-m_1-m_2} B_0^{j-\frac{q}{2}-m_1-m_2, j-m_1} B_1^{j-\frac{q}{2}-m_1, j-\frac{q}{2}} (\mathcal{R}^\perp)^{j-\frac{q}{2}} \\ & \times \left[\frac{2}{3\pi}(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_\alpha}) + \frac{2}{3\pi}(\partial_{\bar{z}_\alpha} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi}(\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, e_{j'}) \right] \\ & \times \left[\frac{4}{3}(\partial_{z_\alpha} R^L)_{x_0}(\partial_{z_{\alpha'}}, \partial_{\bar{z}_{\alpha'}}) + \frac{4}{3}(\partial_{z_{\alpha'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_{\alpha'}}) - \frac{1}{3}(\partial_{\alpha'} R^L)_{x_0}(\partial_{z_\alpha}, e_{\alpha'}) \right]; \end{aligned} \quad (4.110)$$

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_0^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_4 A_2 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
= & \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_0^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} (\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \cdot \left\{ \frac{4}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_l}, \partial_{\bar{z}_l}) \left[\frac{4}{3} (\partial_{z_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) \right. \right. \\
& + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_\alpha}) - \frac{1}{3} (\partial_\alpha R^L)_{x_0} (\partial_{z_r}, e_\alpha) \Big] \\
& + \frac{4}{3\pi} (\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) \left[\frac{4}{3} (\partial_{z_l} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) \right. \\
& \left. \left. + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{z_l}, \partial_{\bar{z}_\alpha}) - \frac{1}{3} (\partial_\alpha R^L)_{x_0} (\partial_{z_l}, e_\alpha) \right] \right\}. \tag{4.111}
\end{aligned}$$

By (2.20) and (4.17), we have

$$(\bar{z}_\alpha z_\beta b_i^+ b_{\alpha'} \mathcal{P})(0, Z) = [(2\delta_{i\alpha'} b_\alpha z_\beta + 4\delta_{i\alpha'} \delta_{\alpha\beta}) \mathcal{P}](0, Z), \tag{4.112}$$

$$(\bar{z}_\alpha z_\beta b_i^+ b_l z_{\alpha'} \bar{z}_r' \mathcal{P})(0, Z) = 0, \tag{4.113}$$

$$(\bar{z}_\alpha z_\beta b_i^+ b_l z_{\alpha'} z_{j'} \mathcal{P})(0, Z) = 0, \tag{4.114}$$

$$\int_{\mathbb{R}^{2n}} (\bar{z}_\alpha \bar{z}_\beta b_i^+ b_{\alpha'} \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0, \tag{4.115}$$

$$\int_{\mathbb{R}^{2n}} (\bar{z}_\alpha \bar{z}_\beta b_i^+ b_l z_{\alpha'} \bar{z}_r' \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0, \tag{4.116}$$

$$\begin{aligned}
(\bar{z}_\alpha \bar{z}_\beta b_i^+ b_l z_{\alpha'} z_{j'} \mathcal{P})(0, Z) &= \frac{\delta_{li}}{\pi^2} (4\delta_{j'\alpha} \delta_{\alpha'\beta} + 2\delta_{j'\alpha} b_\beta z_{\alpha'} + b_\alpha b_\beta z_{\alpha'} z_{j'} \\
&+ 2\delta_{\alpha'\beta} b_\alpha z_{j'} + 2\delta_{\alpha\alpha'} b_\beta z_{j'} + 2\delta_{j'\beta} b_\alpha z_{\alpha'} + 4\delta_{j'\beta} \delta_{\alpha\alpha'}). \tag{4.117}
\end{aligned}$$

By (4.98), (4.100) and (4.112)-(4.117), we have

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2,j-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_1 A_3 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
= & -\frac{8}{3} \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} \\
& \cdot (B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) (\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \times (\partial_{z_\alpha} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_{\alpha'}}) \left[\frac{2}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_{\alpha'}}) \right. \\
& \left. + \frac{2}{3\pi} (\partial_{\bar{z}_{\alpha'}} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi} (\partial_{j'} R^L)_{x_0} (\partial_{\bar{z}_{\alpha'}}, e_{j'}) \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{16}{9\pi^2} \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} \\
& \cdot (B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-m_1,j-\frac{q}{2}} + B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}})(\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \times [(\partial_{\bar{z}_\beta} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, \partial_{z_l})(\partial_{z_\beta} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) \\
& + (\partial_{\bar{z}_\beta} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, \partial_{z_l})(\partial_{z_\alpha} R^L)_{x_0}(\partial_{z_\beta}, \partial_{\bar{z}_l})].
\end{aligned} \tag{4.118}$$

In the following, we write $T = 0$ if

$$\int_{\mathbb{R}^{2n}} T(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0. \tag{4.119}$$

We know that

$$\begin{aligned}
(\bar{z}'_r b_{\alpha'} \mathcal{P})(0, Z) &= (\bar{z}'_r b_l z_{\alpha'} \bar{z}'_{r'} \mathcal{P})(0, Z) = (\bar{z}'_r b_l z_{\alpha} z_{j'} \mathcal{P})(0, Z) \\
&= (z_{j'} b_l z_{\alpha'} \bar{z}'_{r'} \mathcal{P})(0, Z) = (z_{j'} b_l z_{\alpha} z_{j''} \mathcal{P})(0, Z) = 0,
\end{aligned} \tag{4.120}$$

$$(z_{j'} b_{\alpha'} \mathcal{P})(0, Z) = [(b_{\alpha'} z_{j'} + 2\delta_{j'\alpha'}) \mathcal{P}](0, Z), \tag{4.121}$$

so

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_2 A_3 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
&= \frac{2}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} \\
& \cdot (B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}})(\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \times \left[\frac{4}{3} (\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0}(\partial_{z_{j'}}, \partial_{\bar{z}_\alpha}) - \frac{1}{3} (\partial_\alpha R^L)_{x_0}(\partial_{z_{j'}}, e_\alpha) \right] \\
& \times \left[\frac{2}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_{j'}}) + \frac{2}{3\pi} (\partial_{\bar{z}_{j'}} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi} (\partial_{j''} R^L)_{x_0}(\partial_{\bar{z}_{j'}}, e_{j''}) \right].
\end{aligned} \tag{4.122}$$

It holds that

$$\begin{aligned}
b_l b_{l'} \mathcal{P} &= b_l b_{l'} z_{\alpha} \bar{z}'_r \mathcal{P} = b_l z_{\alpha} \bar{z}'_r b_{l'} \mathcal{P} = b_l z_{\alpha} \bar{z}'_r b_{l'} z_{\alpha'} \bar{z}'_{r'} \mathcal{P} \\
&= b_l z_{\alpha} \bar{z}'_r b_{l'} z_{\alpha'} z_{j'} \mathcal{P} = b_l z_{\alpha} z_j b_{l'} z_{\alpha'} \bar{z}'_{r'} \mathcal{P} = b_l z_{\alpha} z_j b_{l'} z_{\alpha'} z_{j'} \mathcal{P} = 0,
\end{aligned} \tag{4.123}$$

$$(b_l z_{\alpha} z_{j'} b_{l'} \mathcal{P})(0, Z) = [(2\delta_{j'l'} b_l z_{\alpha} + 2\delta_{\alpha l'} b_l z_{j'} + b_l b_{l'} z_{\alpha} z_{j'}) \mathcal{P}](0, Z). \tag{4.124}$$

By (4.49), (4.123) and (4.124), we have

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_3^2 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
&= \frac{8}{3} \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2,j-m_1}
\end{aligned}$$

$$\begin{aligned}
& (2B_2^{j-\frac{q}{2}-m_1, j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1, j-\frac{q}{2}})(\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \times \left\{ (\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_{j'}}) \left[\frac{2}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_\alpha}) \right. \right. \\
& + \frac{2}{3\pi} (\partial_{\bar{z}_\alpha} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi} (\partial_{j'} R^L)_{x_0}(\partial_{\bar{z}_\alpha}, e_{j'}) \Big] \\
& + (\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) \left[\frac{2}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_{j'}}) \right. \\
& \left. \left. + \frac{2}{3\pi} (\partial_{\bar{z}_{j'}} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi} (\partial_{j''} R^L)_{x_0}(\partial_{\bar{z}_{j'}}, e_{j''}) \right] \right\} \quad (4.125)
\end{aligned}$$

We know that

$$b_l b_r z_\alpha b_{l'} \mathcal{P} = b_l b_r z_\alpha b_{l'} z_{\alpha'} \bar{z}_{r'} \mathcal{P} = 0, \quad (4.126)$$

$$b_l b_r z_\alpha b_{l'} z_{\alpha'} z_{j'} \mathcal{P} = (2\delta_{\alpha l'} b_l b_r z_{\alpha'} z_{j'} + b_l b_r b_{l'} z_\alpha z_{\alpha'} z_{j'}) \mathcal{P}. \quad (4.127)$$

$$\begin{aligned}
& (b_l b_r b_{l'} z_\alpha z_{\alpha'} z_{j'} \mathcal{P})(0, Z) = -8[\delta_{\alpha l'}(\delta_{l\alpha'} \delta_{j'r} + \delta_{r\alpha'} \delta_{lj'}) \\
& + \delta_{\alpha r}(\delta_{lj'} \delta_{l'\alpha'} + \delta_{l\alpha'} \delta_{j'l'}) + \delta_{\alpha l}(\delta_{r\alpha'} \delta_{l'j'} + \delta_{rj'} \delta_{l'\alpha'})]. \quad (4.128)
\end{aligned}$$

By (4.126)-(4.128), we get

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1, j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2, j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_4 A_3 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
& = \frac{16}{9\pi} \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1, j-\frac{q}{2}-m_1-m_2} B_1^{j-\frac{q}{2}-m_1-m_2, j-\frac{q}{2}-m_1} (\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \times \left\{ (B_2^{j-\frac{q}{2}-m_1, j-\frac{q}{2}} - B_3^{j-\frac{q}{2}-m_1, j-\frac{q}{2}}) \left[(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l})(\partial_{z_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_\alpha}) \right. \right. \\
& + (\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l})(\partial_{z_l} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_\alpha}) \Big] \\
& - B_3^{j-m_1, j} \left[(\partial_{\bar{z}_\alpha} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_{j'}})(\partial_{z_{j'}} R^L)_{x_0}(\partial_{z_{l'}}, \partial_{\bar{z}_{l'}}) \right. \\
& + (\partial_{\bar{z}_\alpha} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l})(\partial_{z_{l'}} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_{l'}}) \\
& + (\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_l})(\partial_{z_{l'}} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_{l'}}) \\
& \left. \left. + (\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_l})(\partial_{z_r} R^L)_{x_0}(\partial_{z_{\alpha'}}, \partial_{\bar{z}_{\alpha'}}) \right] \right\} \quad (4.129)
\end{aligned}$$

It holds that

$$\begin{aligned}
& (\bar{z}_\alpha z_\beta b_i^+ b_l b_r z_{\alpha'} \mathcal{P})(0, Z) = \frac{2}{\pi} [\delta_{ri} (b_l b_\alpha z_\beta z_{\alpha'} + 2\delta_{\alpha\beta} b_l z_{\alpha'} + 2\delta_{\alpha\alpha'} b_l z_\beta + 2\delta_{l\beta} b_\alpha z_{\alpha'} + 4\delta_{l\beta} \delta_{\alpha\alpha'}) \\
& + \delta_{li} (b_r b_\alpha z_\beta z_{\alpha'} + 2\delta_{\alpha\beta} b_r z_{\alpha'} + 2\delta_{\alpha\alpha'} b_r z_\beta + 2\delta_{r\beta} b_\alpha z_{\alpha'} + 4\delta_{r\beta} \delta_{\alpha\alpha'})], \quad (4.130)
\end{aligned}$$

$$\bar{z}_\alpha \bar{z}_r b_i^+ b_l b_{r'} z_{\alpha'} \mathcal{P} = 0. \quad (4.131)$$

By (4.130) and (4.131), we have

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_1 A_4 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
& = -\frac{16}{9\pi^2} \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2,j-m_1} (\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \left[(B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) (\partial_{z_\alpha} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_r}) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_l}, \partial_{\bar{z}_l}) \right. \\
& + (B_2^{j-m_1,j} + B_0^{j-m_1,j} - B_1^{j-m_1,j}) (\partial_{z_l} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_r}) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) \\
& + (B_2^{j-m_1,j} + B_0^{j-m_1,j} - B_1^{j-m_1,j}) (\partial_{z_r} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_l}) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) \\
& + (B_2^{j-m_1,j} - B_1^{j-m_1,j}) (\partial_{z_\alpha} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_l}) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_l}) \\
& - B_1^{j-m_1,j} (\partial_{z_l} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_r}) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) \\
& \left. - B_1^{j-m_1,j} (\partial_{z_r} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_l}) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) \right]. \tag{4.132}
\end{aligned}$$

It holds that

$$\bar{z}_{r'} b_l b_r z_\alpha \mathcal{P} = 0, \tag{4.133}$$

$$z_{j'} b_l b_r z_\alpha \mathcal{P} = (b_l b_r z_{j'} z_\alpha + 2\delta_{j'r'} b_l z_\alpha + 2\delta_{j'l} b_r z_\alpha) \mathcal{P}. \tag{4.134}$$

By (4.133) and (4.134), we get

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1} (\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_2 A_4 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
& = \frac{4}{3\pi} \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} \\
& \cdot (B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) (\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \left\{ (\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) \left[\frac{4}{3} (\partial_{z_l} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{z_l}, \partial_{\bar{z}_\alpha}) \right. \right. \\
& - \frac{1}{3} (\partial_\alpha R^L)_{x_0} (\partial_{z_l}, e_\alpha) \left. \right] + (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_l}, \partial_{\bar{z}_l}) \\
& \left. \times \left[\frac{4}{3} (\partial_{z_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_\alpha}) - \frac{1}{3} (\partial_\alpha R^L)_{x_0} (\partial_{z_r}, e_\alpha) \right] \right\}
\end{aligned} \tag{4.135}$$

We know that

$$b_{l'} b_l b_r z_\alpha \mathcal{P} = b_{l'} z_\alpha \bar{z}_{r'} b_l b_r z_\alpha \mathcal{P} = b_{l'} b_{r'} z_{\alpha'} b_l b_r z_\alpha \mathcal{P} = 0, \tag{4.136}$$

$$\begin{aligned}
& b_{l'} z_{\alpha'} z_{j'} b_l b_r z_\alpha \mathcal{P} = (b_{l'} b_l b_r z_{\alpha'} z_{j'} z_\alpha + 2\delta_{\alpha'r} b_{l'} b_l z_{j'} z_\alpha + 2\delta_{j'r} b_{l'} b_l z_{\alpha'} z_\alpha \\
& + 2\delta_{\alpha'l} b_{l'} b_r z_{j'} z_\alpha + 2\delta_{j'l} b_{l'} b_r z_{\alpha'} z_\alpha + 4\delta_{\alpha'l} \delta_{j'r} b_{l'} z_\alpha + 4\delta_{j'l} \delta_{r\alpha'} b_{l'} z_\alpha) \mathcal{P}. \tag{4.137}
\end{aligned}$$

By (4.136) and (4.137), we get

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \cdot \mathcal{L}_0^{-1}(\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_3 A_4 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) \\
= & \frac{16}{9\pi} \frac{1}{(4\pi)^{j-\frac{q}{2}+2}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{R}^\perp)^{j-\frac{q}{2}} \\
& \cdot \left\{ (-B_3^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} + 2B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) \right. \\
& \times [(\partial_{z_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_\alpha})(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) + (\partial_{z_l} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_\alpha})(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l})] \\
& + (-B_3^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} + B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) [(\partial_{z_l} R^L)_{x_0}(\partial_{z_{l'}}, \partial_{\bar{z}_{l'}})(\partial_{\bar{z}_\alpha} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) \\
& + (\partial_{z_{l'}} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_{l'}})(\partial_{\bar{z}_\alpha} R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_l}) + (\partial_{z_{l'}} R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_{l'}})(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_l}) \\
& \left. + (\partial_{z_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_{l'}})(\partial_{\bar{z}_r} R^L)_{x_0}(\partial_{z_l}, \partial_{\bar{z}_l}) \right\}. \tag{4.138}
\end{aligned}$$

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} \frac{1}{(4\pi)^{j-\frac{q}{2}-m_1+1}} B_0^{1,j-\frac{q}{2}-m_1-m_2} B_2^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \\
& \times \mathcal{L}_0^{-1}(\mathcal{R}^\perp)^{j-\frac{q}{2}-m_1} (A_4^2 \mathcal{P})(0, Z) \mathcal{P}(Z, 0) d\nu_{TX}(Z) = 0. \tag{4.139}
\end{aligned}$$

By the basic assumption, then

$$(\nabla_U^X R^L)(V, W) = \left\langle (\nabla_U^X \tilde{J})V, W \right\rangle. \tag{4.140}$$

Direct computations show that

$$(b_l z_\alpha + 2\delta_{\alpha l})(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) z_{j'} \mathcal{P} = \frac{\delta_{rj'}}{\pi} (b_l z_\alpha + 2\delta_{\alpha l}) \mathcal{P}; \tag{4.141}$$

$$(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) z_{j'} \mathcal{P} = \frac{\delta_{rj'}}{\pi} \mathcal{P}; \tag{4.142}$$

$$(b_l z_\alpha + 2\delta_{\alpha l})(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) \bar{z}_{j'} \mathcal{P} = (\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) \bar{z}_{j'} \mathcal{P} = 0; \tag{4.143}$$

$$(b_l z_\alpha + 2\delta_{\alpha l})(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) b_{j'} \mathcal{P} = (\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) b_{j'} \mathcal{P} = 0; \tag{4.144}$$

$$(b_l z_\alpha + 2\delta_{\alpha l})(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) b_{l'} z_{\alpha'} \bar{z}'_{r'} \mathcal{P} = \left[\frac{b_l b_{l'} z_\alpha \bar{z}'_{r'}}{\pi} \delta_{rr'} + \frac{2\delta_{\alpha l'} \delta_{rr'}}{\pi} b_l \bar{z}'_{r'} + \frac{2\delta_{\alpha l} \delta_{rr'}}{\pi} b_{l'} \bar{z}'_{r'} \right] \mathcal{P}; \tag{4.145}$$

$$(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) b_{l'} z_{\alpha'} \bar{z}'_{r'} \mathcal{P} = \frac{\delta_{rr'}}{\pi} b_{l'} \bar{z}'_{r'} \mathcal{P}; \tag{4.146}$$

$$\begin{aligned}
& (b_l z_\alpha + 2\delta_{\alpha l})(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r) b_{l'} z_{\alpha'} z_{j'} \mathcal{P} = \\
& \left[\frac{b_l b_{l'} z_\alpha z_{\alpha'}}{\pi} \delta_{rj'} + \frac{2\delta_{\alpha l'} \delta_{rj'}}{\pi} b_l z_{\alpha'} + \frac{b_l b_{l'} z_\alpha z_{j'}}{\pi} \delta_{r\alpha'} \right]
\end{aligned}$$

$$+\frac{2\delta_{\alpha l'}\delta_{r\alpha'}}{\pi}b_l z_{j'}+\frac{2\delta_{\alpha l}\delta_{rj'}}{\pi}b_{l'} z_{\alpha'}+\frac{2\delta_{\alpha l}\delta_{r\alpha'}}{\pi}b_{l'} z_{j'}]\mathcal{P}; \quad (4.147)$$

$$(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r)b_{l'} z_{\alpha'} z_{j'} \mathcal{P} = [\frac{\delta_{rj'}}{\pi}b_{l'} z_{\alpha'} + \frac{\delta_{r\alpha'}}{\pi}b_{l'} z_{j'}]\mathcal{P}; \quad (4.148)$$

$$(b_l z_{\alpha} + 2\delta_{\alpha l})(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r)b_{l'} b_{r'} z_{\alpha'} \mathcal{P} = [\frac{\delta_{r\alpha'}}{\pi}b_l b_{l'} b_{r'} z_{\alpha} \\ + \frac{2\delta_{\alpha r'}\delta_{r\alpha'}}{\pi}b_l b_{l'} + \frac{2\delta_{\alpha l'}\delta_{r\alpha'}}{\pi}b_l b_{r'} + \frac{2\delta_{\alpha l}\delta_{r\alpha'}}{\pi}b_{l'} b_{r'}]\mathcal{P}; \quad (4.149)$$

$$(\bar{z}_r - \frac{b_r}{2\pi} - \bar{z}'_r)b_{l'} b_{r'} z_{\alpha'} \mathcal{P} = \frac{\delta_{r\alpha'}}{\pi}b_{l'} b_{r'} \mathcal{P}. \quad (4.150)$$

By (4.96),(4.102) and (4.141)-(4.150), we get

$$\begin{aligned} & \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_1}(\mathcal{L}_0^{-1}P^{N^\perp}A_5)(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_2} \\ & \times (\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}'_1)(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}-m_1-m_2}P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})^{j-\frac{q}{2}}I_{2j} \\ & = \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}} \frac{1}{(4\pi)^{2j-q+2}}B_0^{1,j-\frac{q}{2}}B_0^{1,j-\frac{q}{2}-m_1-m_2}B_0^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1}(\mathcal{R}^\perp)^{j-\frac{q}{2}} \\ & \left\{ (B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}})\frac{4}{3\pi}(\partial_{\bar{z}_r}R^L)_{x_0}(\partial_{z_\alpha},\partial_{\bar{z}_\alpha}) \right. \\ & \cdot [\frac{4}{3}(\partial_{z_r}R^L)_{x_0}(\partial_{z_{\alpha'}},\partial_{\bar{z}_{\alpha'}}) + \frac{4}{3}(\partial_{z_{\alpha'}}R^L)_{x_0}(\partial_{z_r},\partial_{\bar{z}_{\alpha'}}) - \frac{1}{3}(\partial_{\alpha'}R^L)_{x_0}(\partial_{z_r},e_{\alpha'})] \\ & - \frac{1}{3\pi}B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}(\partial_{j'}R^L)_{x_0}(\partial_{\bar{z}_r},e_{j'})[\frac{4}{3}(\partial_{z_r}R^L)_{x_0}(\partial_{z_{\alpha'}},\partial_{\bar{z}_{\alpha'}}) \\ & + \frac{4}{3}(\partial_{z_{\alpha'}}R^L)_{x_0}(\partial_{z_r},\partial_{\bar{z}_{\alpha'}}) - \frac{1}{3}(\partial_{\alpha'}R^L)_{x_0}(\partial_{z_r},e_{\alpha'})] \Big\} I_{\det(\overline{W}^*) \otimes E}(\mathcal{R}^{\perp,*})^{j-\frac{q}{2}} \\ & + \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}} \frac{1}{(4\pi)^{2j-q+2}}B_0^{1,j-\frac{q}{2}}B_0^{1,j-\frac{q}{2}-m_1-m_2}B_1^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1}(\mathcal{R}^\perp)^{j-\frac{q}{2}} \\ & \left\{ \frac{16}{9\pi}(B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}})[(\partial_{\bar{z}_r}R^L)_{x_0}(\partial_{z_\alpha},\partial_{\bar{z}_\alpha})(\partial_{z_r}R^L)_{x_0}(\partial_{z_{\alpha'}},\partial_{\bar{z}_{\alpha'}}) \right. \\ & + (\partial_{\bar{z}_r}R^L)_{x_0}(\partial_{z_\alpha},\partial_{\bar{z}_l})(\partial_{z_r}R^L)_{x_0}(\partial_{z_l},\partial_{\bar{z}_\alpha}) + (\partial_{\bar{z}_r}R^L)_{x_0}(\partial_{z_\alpha},\partial_{\bar{z}_\alpha})(\partial_{z_{l'}}R^L)_{x_0}(\partial_{z_r},\partial_{\bar{z}_{l'}}) \\ & + (\partial_{\bar{z}_r}R^L)_{x_0}(\partial_{z_\alpha},\partial_{\bar{z}_l})(\partial_{z_l}R^L)_{x_0}(\partial_{z_r},\partial_{\bar{z}_\alpha})] \\ & + \frac{4}{9\pi}B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}[(\partial_{j'}R^L)_{x_0}(\partial_{\bar{z}_r},e_{j'})(\partial_{z_r}R^L)_{x_0}(\partial_{z_{\alpha'}},\partial_{\bar{z}_{\alpha'}}) \\ & + (\partial_{j'}R^L)_{x_0}(\partial_{\bar{z}_r},e_{j'})(\partial_{z_{l'}}R^L)_{x_0}(\partial_{z_r},\partial_{\bar{z}_{l'}})] \Big\} I_{\det(\overline{W}^*) \otimes E}(\mathcal{R}^{\perp,*})^{j-\frac{q}{2}}. \quad (4.151) \end{aligned}$$

Write $B_{l_1}B_{l_2}B_{l_3}$ for $B_{l_1}^{1,j-\frac{q}{2}-m_1-m_2}B_{l_2}^{j-\frac{q}{2}-m_1-m_2,j-\frac{q}{2}-m_1}B_{l_3}^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}$. By (4.96)-(4.101),(4.108)-(4.111),(4.118),(4.112),(4.125),(4.129),(4.132),(4.135),(4.138) and (4.139), we get

$$\begin{aligned}
& \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} (A_1 + A_2 + A_3 + A_4)) \\
& (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1 - m_2} P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j - \frac{q}{2}} I_{2j} \\
= & \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2}} \frac{1}{2^{j - \frac{q}{2}} (j - \frac{q}{2})! (4\pi)^{2j - q + 2}} I_{2j} \left\{ \frac{1}{9\pi} (B_0 B_0 B_2 - B_0 B_0 B_1) \right. \\
& \times (2A_{\bar{l}l s} A_{sr\bar{r}} - 2A_{\bar{l}l s} A_{r\bar{r}s} + 2A_{\bar{s}l\bar{l}} A_{sr\bar{r}} \\
& - 2A_{\bar{s}l\bar{l}} A_{r\bar{r}s} - A_{\bar{l}l\bar{s}} A_{e_r s e_r} - A_{\bar{s}l\bar{l}} A_{e_r s e_r}) \\
& + [-\frac{4}{9\pi} B_0 B_1 B_0 - \frac{2}{3\pi} B_0 B_1 B_1 - \frac{4}{9\pi} B_0 B_1 B_2 \\
& + \frac{2}{9\pi} B_0 B_1 B_3 - (\frac{2}{9\pi^2} + \frac{4}{9\pi}) B_0 B_2 B_2 - (\frac{2}{9\pi^2} + \frac{2}{9\pi}) B_0 B_2 B_1 \\
& + \frac{2}{9\pi} B_0 B_2 B_3] (A_{\bar{l}l s} A_{\bar{r}r\bar{s}} + A_{\bar{l}l s} A_{\bar{s}r\bar{r}}) \\
& + [(-\frac{2}{9\pi^2} - \frac{2}{9\pi}) B_0 B_1 B_2 + \frac{2}{9\pi^2} B_0 B_1 B_1 - \frac{2}{9\pi^2} B_0 B_1 B_0 \\
& + \frac{2}{9\pi} B_0 B_1 B_3 + (-\frac{2}{9\pi^2} - \frac{4}{9\pi}) B_0 B_2 B_2 - \frac{2}{9\pi^2} B_0 B_2 B_0 \\
& + (\frac{4}{9\pi^2} + \frac{2}{9\pi}) B_0 B_2 B_1 + \frac{2}{9\pi} B_0 B_2 B_3] (A_{\bar{l}sr} A_{l s \bar{r}} + A_{\bar{l}sr} A_{s l \bar{r}}) \\
& + \frac{1}{9\pi} (2B_0 B_1 B_0 - 4B_0 B_1 B_1 + 4B_0 B_1 B_2 - 2B_0 B_1 B_3 \\
& - 2B_0 B_2 B_1 - 2B_0 B_2 B_3 + 4B_0 B_2 B_2) (A_{s l \bar{l}} A_{\bar{r}r\bar{s}} + A_{s l \bar{l}} A_{\bar{s}r\bar{r}}) \\
& + \frac{1}{9\pi} (-B_0 B_1 B_0 + B_0 B_1 B_1 - B_0 B_2 B_0 \\
& + B_0 B_2 B_1) (A_{e_r s e_r} A_{\bar{r}r\bar{s}} + A_{e_r s e_r} A_{\bar{s}r\bar{r}}) \\
& + \frac{1}{9\pi} (2B_0 B_1 B_0 - 3B_0 B_1 B_1 + 2B_0 B_1 B_2 - B_0 B_0 B_1) A_{e_r \bar{s} e_r} A_{\bar{l}l s} \\
& + \frac{1}{9\pi} (-B_0 B_1 B_0 + 2B_0 B_1 B_1 + 2B_0 B_1 B_2 + B_0 B_0 B_1) A_{e_r \bar{s} e_r} A_{s l \bar{l}} \\
& + \frac{1}{18\pi} (B_0 B_1 B_0 - B_0 B_1 B_1 - B_0 B_0 B_1) A_{e_r s e_r} A_{e_l \bar{s} e_l} \Big\} \\
& \cdot (\mathcal{R}^\perp)(x)^{j - \frac{q}{2}} I_{\det(\bar{W}^*) \otimes E} (\mathcal{R}^{\perp,*})(x)^{j - \frac{q}{2}} I_{2j}. \tag{4.152}
\end{aligned}$$

By (4.97), then

$$V_a^0 = (4.151) + (4.152). \tag{4.153}$$

Set

$$\begin{aligned}
V_a^1 = & \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} \\
& \cdot (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1 - m_2} P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j - \frac{q}{2}} I_{2j}. \tag{4.154}
\end{aligned}$$

Set $n = 4, j = 2, q = 2$. By (4.89) and $(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N$ and $(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})P^N$ being $(0, (2, 2))$ forms, then

$$\begin{aligned} V_a^1 &= I_4[(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'') \\ &\quad + (\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'') \\ &\quad + (\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})]P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})I_4 \\ &= I_4(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})I_4. \end{aligned} \quad (4.155)$$

Let

$$\widehat{\Phi}(U) = \sum_{l \leq q} \sum_{q+1 \leq m} \langle (\nabla_U^B \mathbf{J})_{x_0} w_l, \overline{w}_m \rangle \overline{w}^m \wedge i_{\overline{w}_l}, \quad \text{for } U \in TX. \quad (4.156)$$

$$\widetilde{\Phi}(U) = \sum_{m \leq q} \sum_{q+1 \leq l} \langle (\nabla_U^B \mathbf{J})_{x_0} w_l, \overline{w}_m \rangle \overline{w}^m \wedge i_{\overline{w}_l}, \quad \text{for } U \in TX. \quad (4.157)$$

By (4.89) and (4.156), then

$$(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N = -\frac{\sqrt{-1}}{2}\widehat{\Phi}(\partial_{z_s})z_s - \frac{\sqrt{-1}}{2}\widehat{\Phi}(\partial_{\overline{z}_s})\overline{z}_s - \frac{\sqrt{-1}}{6\pi}\widehat{\Phi}(\partial_{\overline{z}_s})b_s. \quad (4.158)$$

By (4.89) and (4.156)-(4.158), similar to the computations of V_a^0 , we get

$$\begin{aligned} V_a^1 &= \left[-\frac{7}{96 \times 120\pi^3} \mathcal{R}^\top \widehat{\Phi}(\partial_{z_s}) \widehat{\Phi}(\partial_{\overline{z}_s}) + \frac{1}{288\pi^3} \mathcal{R}^\perp \widetilde{\Phi}(\partial_{z_s}) \widehat{\Phi}(\partial_{\overline{z}_s}) \right. \\ &\quad \left. - \frac{7}{32 \times 120\pi^3} \mathcal{R}^\top \widehat{\Phi}(\partial_{\overline{z}_s}) \mathcal{R}^\top \widehat{\Phi}(\partial_{z_s}) + \frac{1}{96\pi^3} \mathcal{R}^\perp \widetilde{\Phi}(\partial_{\overline{z}_s}) \widehat{\Phi}(\partial_{z_s}) \right] I_{\det(\overline{W}^*) \otimes E}(\mathcal{R}^{\perp,*})(x)I_4. \end{aligned} \quad (4.159)$$

Set

$$\begin{aligned} V_a^2 &= \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_1}(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_2} \\ &\quad \cdot (\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1 - m_2} P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})^{j - \frac{q}{2}} I_{2j}. \end{aligned} \quad (4.160)$$

By $n = 4, j = 2, q = 2$, then

$$\begin{aligned} V_a^2 &= I_4[(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'') \\ &\quad + (\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')]P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})I_4. \end{aligned} \quad (4.161)$$

By (4.158), similar to the computations of V_a^0 , then

$$\begin{aligned} &(\mathcal{L}_0^{-1}P^{N^\perp}A_1)(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N(0, Z) \\ &= \frac{\sqrt{-1}}{648\pi^3}(\partial_{z_\alpha}R^L)_{x_0}(\partial_{\overline{z}_\alpha}, \partial_{z_s})\mathcal{R}^0\widehat{\Phi}(\partial_{\overline{z}_s})I_{\det(\overline{W}^*) \otimes E}P^N(0, Z); \\ &(\mathcal{L}_0^{-1}P^{N^\perp}A_2)(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N(0, Z) \end{aligned} \quad (4.162)$$

$$\begin{aligned}
&= \frac{-\sqrt{-1}}{864\pi^3} \left[\frac{4}{3} (\partial_{z_s} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{z_s}, \partial_{\bar{z}_\alpha}) \right. \\
&\quad \left. - \frac{1}{3} (\partial_\alpha R^L)_{x_0} (\partial_{z_s}, e_\alpha) \right] \mathcal{R}^0 \widehat{\Phi}(\partial_{\bar{z}_s}) I_{\det(\overline{W}^*) \otimes E} P^N(0, Z); \tag{4.163}
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{L}_0^{-1} P^{N^\perp} A_3)(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})(\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') P^N(0, Z) \\
&= \frac{\sqrt{-1}}{96\pi^2} \left[\frac{2}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_\alpha}) + \frac{2}{3\pi} (\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_r}) \right. \\
&\quad \left. - \frac{1}{6\pi} (\partial_{j'} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, e_{j'}) \right] \mathcal{R}^0 \widehat{\Phi}(\partial_{z_\alpha}) I_{\det(\overline{W}^*) \otimes E} \\
&\quad + \frac{\sqrt{-1}}{1296\pi^3} \left[(\partial_{z_s} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) \mathcal{R}^0 \widehat{\Phi}(\partial_{\bar{z}_s}) \right. \\
&\quad \left. + (\partial_{z_{l'}} R^L)_{x_0} (\partial_{z_s}, \partial_{\bar{z}_{l'}}) \mathcal{R}^0 \widehat{\Phi}(\partial_{\bar{z}_s}) \right] I_{\det(\overline{W}^*) \otimes E} P^N(0, Z); \tag{4.164}
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{L}_0^{-1} P^{N^\perp} A_4)(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})(\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') P^N(0, Z) \\
&= \frac{-\sqrt{-1}}{192\pi^3} \left[(\partial_{\bar{z}_s} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) \mathcal{R}^0 \widehat{\Phi}(\partial_{z_s}) \right. \\
&\quad \left. + (\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_s}) \mathcal{R}^0 \widehat{\Phi}(\partial_{z_s}) \right] I_{\det(\overline{W}^*) \otimes E} P^N(0, Z); \tag{4.165}
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{L}_0^{-1} P^{N^\perp} A_5)(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})(\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') P^N(0, Z) \\
&= \frac{-\sqrt{-1}}{96\pi^3} \left[\frac{1}{3} (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) \mathcal{R}^0 \widehat{\Phi}(\partial_{z_r}) \right. \\
&\quad \left. - \frac{1}{4} (\partial_{j'} R^L)_{x_0} (\partial_{\bar{z}_r}, e_{j'}) \mathcal{R}^0 \widehat{\Phi}(\partial_{z_r}) \right] P^N(0, Z). \tag{4.166}
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})(\mathcal{L}_0^{-1} P^{N^\perp} A_1)(\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') P^N(0, Z) \\
&= \frac{5\sqrt{-1}}{1296\pi^3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_s}) \mathcal{R}^0 \widehat{\Phi}(\partial_{\bar{z}_s}) I_{\det(\overline{W}^*) \otimes E} P^N(0, Z); \tag{4.167}
\end{aligned}$$

$$\begin{aligned}
&(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})(\mathcal{L}_0^{-1} P^{N^\perp} A_2)(\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') P^N(0, Z) \\
&= \frac{-\sqrt{-1}}{1728\pi^3} \left[\frac{4}{3} (\partial_{z_s} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{z_s}, \partial_{\bar{z}_\alpha}) \right. \\
&\quad \left. - \frac{1}{3} (\partial_\alpha R^L)_{x_0} (\partial_{z_s}, e_\alpha) \right] \mathcal{R}^0 \widehat{\Phi}(\partial_{\bar{z}_s}) I_{\det(\overline{W}^*) \otimes E} P^N(0, Z); \tag{4.168}
\end{aligned}$$

$$\begin{aligned}
& (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}A_3)(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N(0, Z) \\
&= \frac{\sqrt{-1}}{144\pi^2}[\frac{2}{3\pi}(\partial_{\bar{z}_r}R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_\alpha}) + \frac{2}{3\pi}(\partial_{\bar{z}_\alpha}R^L)_{x_0}(\partial_{z_r}, \partial_{\bar{z}_r}) \\
&\quad - \frac{1}{6\pi}(\partial_{j'}R^L)_{x_0}(\partial_{\bar{z}_\alpha}, e_{j'})]\mathcal{R}^0\widehat{\Phi}(\partial_{z_\alpha})I_{\det(\overline{W}^*)\otimes E} \\
&\quad + \frac{7\sqrt{-1}}{36 \times 144\pi^3}[(\partial_{z_s}R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha})\mathcal{R}^0\widehat{\Phi}(\partial_{\bar{z}_s}) \\
&\quad + (\partial_{z_{l'}}R^L)_{x_0}(\partial_{z_s}, \partial_{\bar{z}_{l'}})\mathcal{R}^0\widehat{\Phi}(\partial_{\bar{z}_s})]I_{\det(\overline{W}^*)\otimes E}P^N(0, Z); \tag{4.169}
\end{aligned}$$

$$\begin{aligned}
& (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}A_4)(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N(0, Z) \\
&= \frac{-\sqrt{-1}}{384\pi^3}[(\partial_{\bar{z}_s}R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha})\mathcal{R}^0\widehat{\Phi}(\partial_{z_s}) \\
&\quad + (\partial_{\bar{z}_\alpha}R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_s})\mathcal{R}^0\widehat{\Phi}(\partial_{z_s})]I_{\det(\overline{W}^*)\otimes E}P^N(0, Z); \tag{4.170}
\end{aligned}$$

$$\begin{aligned}
& (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}A_5)(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')P^N(0, Z) \\
&= \frac{-\sqrt{-1}}{96\pi^3}[\frac{5}{9}(\partial_{\bar{z}_r}R^L)_{x_0}(\partial_{z_\alpha}, \partial_{\bar{z}_\alpha})\mathcal{R}^0\widehat{\Phi}(\partial_{z_r}) \\
&\quad - \frac{1}{4}(\partial_{j'}R^L)_{x_0}(\partial_{\bar{z}_r}, e_{j'})\mathcal{R}^0\widehat{\Phi}(\partial_{z_r})]P^N(0, Z). \tag{4.171}
\end{aligned}$$

By (4.97) and (4.161), then

$$V_a^2 = \frac{1}{4\pi}[(4.162) + \cdots + (4.171)] \times \mathcal{R}^{\perp,*}I_4. \tag{4.172}$$

Set

$$\begin{aligned}
V_a^3 &= \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_1}(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_2} \\
&\quad \cdot (\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1')(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1 - m_2}P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})^{j - \frac{q}{2}}I_{2j}. \tag{4.173}
\end{aligned}$$

By $n = 4$, $j = 2$, $q = 2$, then

$$V_a^3 = I_4(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1'')(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1')P^N(\mathcal{O}_2^{-2}\mathcal{L}_0^{-1})I_4. \tag{4.174}$$

By (4.89) and $(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1')P^N$ being a $(0, (2, 0))$ -form and $\mathcal{L}_0A_2P^N = 0$, then

$$\mathcal{O}_1''(\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1')P^N = -4\pi\sqrt{-1}[\widehat{\Phi}(\partial_{z_s})z_s + \widehat{\Phi}(\partial_{\bar{z}_s})\bar{z}_s](\frac{1}{4\pi}A_3 + \frac{1}{8\pi}A_4)P^N. \tag{4.175}$$

Direct computations show that

$$z_sb_\alpha = b_\alpha z_s + 2\delta_{\alpha s}, \quad z_sb_l z_\alpha \bar{z}_r' = 2\delta_{sl} z_\alpha \bar{z}_r' + b_l z_s z_\alpha \bar{z}_r'; \tag{4.176}$$

$$z_s b_l z_\alpha z_{j'} = 2\delta_{sl} z_\alpha z_{j'} + b_l z_s z_\alpha z_{j'}, \quad z_s b_l b_r z_\alpha = 2\delta_{sl} b_r z_\alpha + 2\delta_{sr} b_l z_\alpha + b_l b_r z_s z_\alpha; \quad (4.177)$$

$$\bar{z}_s b_\alpha \mathcal{P} = \left(\frac{b_\alpha b_s}{2\pi} + b_\alpha \bar{z}'_s \right) \mathcal{P}; \quad (4.178)$$

$$\bar{z}_s b_l z_\alpha \bar{z}'_r \mathcal{P} = \left[\frac{1}{2\pi} b_l b_s z_\alpha \bar{z}'_r + \frac{\delta_{\alpha s}}{\pi} b_l \bar{z}'_r + b_l z_\alpha \bar{z}'_r \bar{z}'_s \right] \mathcal{P}; \quad (4.179)$$

$$\bar{z}_s b_l z_\alpha z_{j'} \mathcal{P} = \left[\frac{b_l b_s z_\alpha z_{j'}}{2\pi} + \frac{\delta_{\alpha s}}{\pi} b_l z_{j'} + \frac{\delta_{j's}}{\pi} b_l z_\alpha + b_l z_\alpha z_{j'} \bar{z}'_s \right] \mathcal{P}; \quad (4.180)$$

$$\bar{z}_s b_l b_r z_\alpha \mathcal{P} = \left[\frac{b_l b_r b_s z_\alpha}{2\pi} + \frac{\delta_{\alpha s}}{\pi} b_l z_{j'} + \frac{\delta_{j's}}{\pi} b_l z_\alpha + b_l z_\alpha z_{j'} \bar{z}'_s \right] \mathcal{P}; \quad (4.181)$$

By (4.174)-(4.181), we can get

$$\begin{aligned} V_a^3 = & \frac{-\sqrt{-1}}{4\pi} \mathcal{R}^0 \left\{ \frac{11}{8 \times 144\pi^3} \widehat{\Phi}(\partial_{z_s}) [(\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_s}) + (\partial_{\bar{z}_s} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha})] \right. \\ & - \frac{5}{6 \times 288\pi^3} \widehat{\Phi}(\partial_{z_s}) (\partial_{j'} R^L)_{x_0} (\partial_{\bar{z}_s}, e_{j'}) - \frac{7}{12 \times 144\pi^3} \widehat{\Phi}(\partial_{\bar{z}_s}) [(\partial_{z_s} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) \\ & \left. + (\partial_{z_l} R^L)_{x_0} (\partial_{z_s}, \partial_{\bar{z}_l})] \right\} \mathcal{R}^{\perp,*} I_4. \end{aligned} \quad (4.182)$$

By (4.84),(4.90),(4.154),(4.160) and (4.173), we get

$$V_a = V_a^0 + V_a^1 + V_a^2 + V_a^3. \quad (4.183)$$

Nextly, we compute V_b . Set

$$\begin{aligned} V_b^0 = & \sum_{0 \leq m_1, m_2 \leq j - \frac{q}{2}} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}'_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1} P^N \\ & \times (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}'_1 \mathcal{L}_0^{-1} P^{N^\perp}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j - \frac{q}{2} - m_2} I_{2j}. \end{aligned} \quad (4.184)$$

By (4.184), then

$$V_b^0 = I_{2j} C C^* I_{2j}, \quad C = \sum_{0 \leq m_1 \leq j - \frac{q}{2}} C_{m_1}, \quad (4.185)$$

where

$$C_{m_1} = (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}'_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1} P^N. \quad (4.186)$$

Similar to (4.96), then

$$\begin{aligned} & C_{m_1}(0, Z) \\ = & \frac{1}{(4\pi)^{j - \frac{q}{2} + 1}} (\mathcal{R}^\perp)^{j - \frac{q}{2}} \left\{ \left(\frac{4}{3} B_0^{1, j - \frac{q}{2} - m_1} B_0^{j - \frac{q}{2} - m_1, j - \frac{q}{2}} \right. \right. \\ & - \frac{8}{3} B_0^{1, j - \frac{q}{2} - m_1} B_1^{j - \frac{q}{2} - m_1, j - \frac{q}{2}} + \frac{4}{3} B_0^{1, j - \frac{q}{2} - m_1} B_2^{j - \frac{q}{2} - m_1, j - \frac{q}{2}} \Big) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_l}, \partial_{\bar{z}_l}) \\ & + \left(\frac{4}{3} B_0^{1, j - \frac{q}{2} - m_1} B_2^{j - \frac{q}{2} - m_1, j - \frac{q}{2}} - \frac{4}{3} B_0^{1, j - \frac{q}{2} - m_1} B_1^{j - \frac{q}{2} - m_1, j - \frac{q}{2}} \right) (\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_r}) \\ & + \left(\frac{1}{3} B_0^{1, j - \frac{q}{2} - m_1} B_1^{j - \frac{q}{2} - m_1, j - \frac{q}{2}} - \frac{1}{3} B_0^{1, j - \frac{q}{2} - m_1} B_0^{j - \frac{q}{2} - m_1, j - \frac{q}{2}} \right) \\ & \left. \cdot (\partial_l R^L)_{x_0} (\partial_{\bar{z}_r}, e_l) \right\} \bar{z}_r [\mathcal{P}(0, Z)]. \end{aligned} \quad (4.187)$$

So

$$\begin{aligned}
& (C_{m_1}^*)(Z, 0) \\
= & -\frac{1}{(4\pi)^{j-\frac{q}{2}+1}} \left\{ \left(\frac{4}{3} B_0^{1,j-\frac{q}{2}-m_1} B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - \frac{8}{3} B_0^{1,j-\frac{q}{2}-m_1} B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} \right. \right. \\
& + \frac{4}{3} B_0^{1,j-\frac{q}{2}-m_1} B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} \left. \right) (\partial_{z_r} R^L)_{x_0} (\partial_{\bar{z}_l}, \partial_{z_l}) \\
& + \left(\frac{4}{3} B_0^{1,j-\frac{q}{2}-m_1} B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - \frac{4}{3} B_0^{1,j-\frac{q}{2}-m_1} B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} \right) (\partial_{z_\alpha} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_r}) \\
& + \left(\frac{1}{3} B_0^{1,j-\frac{q}{2}-m_1} B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - \frac{1}{3} B_0^{1,j-\frac{q}{2}-m_1} B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} \right) \\
& \cdot (\partial_l R^L)_{x_0} (\partial_{z_r}, e_l) \left. \right\} (\mathcal{R}^{\perp,*})^{j-\frac{q}{2}} z_r [\mathcal{P}(Z, 0)]. \tag{4.188}
\end{aligned}$$

By (4.24), (4.185)-(4.188), we get

$$\begin{aligned}
V_b^0 = & \frac{-2}{9 \times (4\pi)^{2j-q+3}} I_{2j} \sum_{0 \leq m_1, m_2 \leq j-\frac{q}{2}} (B_0^{1,j-\frac{q}{2}-m_1})^2 \left[(B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} \right. \\
& - 2B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} + B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) A_{\bar{r}s\bar{s}} \\
& + (B_2^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) A_{\bar{s}s\bar{r}} \\
& + \frac{1}{2} (B_1^{j-\frac{q}{2}-m_1,j-\frac{q}{2}} - B_0^{j-\frac{q}{2}-m_1,j-\frac{q}{2}}) A_{e_l \bar{r} e_l} \left. \right] \\
& \times \left[(B_0^{j-\frac{q}{2}-m_2,j-\frac{q}{2}} - 2B_1^{j-\frac{q}{2}-m_2,j-\frac{q}{2}} + B_2^{j-\frac{q}{2}-m_2,j-\frac{q}{2}}) A_{r s' \bar{s}'} \right. \\
& + (B_2^{j-\frac{q}{2}-m_2,j-\frac{q}{2}} - B_1^{j-\frac{q}{2}-m_2,j-\frac{q}{2}}) A_{s' \bar{s}' r} \\
& + \frac{1}{2} (B_1^{j-\frac{q}{2}-m_2,j-\frac{q}{2}} - B_0^{j-\frac{q}{2}-m_2,j-\frac{q}{2}}) A_{e_{l'} r e_{l'}} \left. \right] (\mathcal{R}^\perp)(x)^{j-\frac{q}{2}} (\mathcal{R}^{\perp,*})(x)^{j-\frac{q}{2}} I_{2j}. \tag{4.189}
\end{aligned}$$

When $n = 4$, $j = 2$ and $q = 2$, by (4.158), then

$$\begin{aligned}
& \sum_{0 \leq m_1 \leq j-\frac{q}{2}} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-m_1} P^N(0, Z) \\
= & I_4 (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') P^N(0, Z) = \frac{-5\sqrt{-1}}{144\pi} \mathcal{R}^0 \hat{\Phi}(\partial_{\bar{z}_s}) \bar{z}_s P^N(0, Z). \tag{4.190}
\end{aligned}$$

Set

$$\begin{aligned}
V_b^1 = & \sum_{0 \leq m_1, m_2 \leq j-\frac{q}{2}} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-m_1} P^N \\
& \times (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1' \mathcal{L}_0^{-1} P^{N^\perp}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-m_2} I_{2j}. \tag{4.191}
\end{aligned}$$

$$\begin{aligned}
V_b^2 = & \sum_{0 \leq m_1, m_2 \leq j-\frac{q}{2}} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-m_1} P^N \\
& \times (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1'' \mathcal{L}_0^{-1} P^{N^\perp}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-m_2} I_{2j}. \tag{4.192}
\end{aligned}$$

$$\begin{aligned}
V_b^3 = & \sum_{0 \leq m_1, m_2 \leq j - \frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} P^{N^\perp} \mathcal{O}_1'') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2} - m_1} P^N \\
& \times (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1'' \mathcal{L}_0^{-1} P^{N^\perp}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j - \frac{q}{2} - m_2} I_{2j}.
\end{aligned} \tag{4.193}$$

By (4.187), (4.188) and (4.190), then

$$\begin{aligned}
V_b^1 = & \frac{5\sqrt{-1}}{144 \times 16\pi^3} \mathcal{R}^0 \widehat{\Phi}(\partial_{\bar{z}_r}) \left\{ \left(\frac{4}{3} B_0^{1,1} B_0^{1,1} - \frac{8}{3} B_0^{1,1} B_1^{1,1} \right. \right. \\
& + \frac{4}{3} B_0^{1,1} B_2^{1,1} \Big) (\partial_{z_r} R^L)_{x_0} (\partial_{\bar{z}_l}, \partial_{z_l}) \\
& + \left(\frac{4}{3} B_0^{1,1} B_2^{1,1} - \frac{4}{3} B_0^{1,1} B_1^{1,1} \right) (\partial_{z_\alpha} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_r}) \\
& + \left(\frac{1}{3} B_0^{1,1} B_1^{1,1} - \frac{1}{3} B_0^{1,1} B_0^{1,1} \right) \\
& \cdot (\partial_l R^L)_{x_0} (\partial_{z_r}, e_l) \Big\} I_{\det(\overline{W}^*) \otimes E} \mathcal{R}^{\perp,*} I_4
\end{aligned} \tag{4.194}$$

$$\begin{aligned}
V_b^2 = & \frac{5\sqrt{-1}}{144 \times 16\pi^3} \left\{ \left(\frac{4}{3} B_0^{1,1} B_0^{1,1} \right. \right. \\
& - \frac{8}{3} B_0^{1,1} B_1^{1,1} + \frac{4}{3} B_0^{1,1} B_2^{1,1} \Big) (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_l}, \partial_{\bar{z}_l}) \\
& + \left(\frac{4}{3} B_0^{1,1} B_2^{1,1} - \frac{4}{3} B_0^{1,1} B_1^{1,1} \right) (\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{z_r}) \\
& + \left(\frac{1}{3} B_0^{1,1} B_1^{1,1} - \frac{1}{3} B_0^{1,1} B_0^{1,1} \right) \\
& \cdot (\partial_l R^L)_{x_0} (\partial_{\bar{z}_r}, e_l) \Big\} \mathcal{R}^\perp I_{\det(\overline{W}^*) \otimes E} \widehat{\Phi}(\partial_{z_r})^* \mathcal{R}^{0,*} I_4
\end{aligned} \tag{4.195}$$

$$V_b^3 = \frac{25}{(144\pi)^2} \mathcal{R}^0 \widehat{\Phi}(\partial_{\bar{z}_r}) I_{\det(\overline{W}^*) \otimes E} \widehat{\Phi}^*(\partial_{z_r}) \mathcal{R}^{0,*} I_4. \tag{4.196}$$

By (4.85), (4.184), (4.191)-(4.193), we get

$$V_b = V_b^0 + V_b^1 + V_b^2 + V_b^3. \tag{4.197}$$

4.4 The computations of the term IV

In this section, we will compute the term IV. By the discussions after (4.2), for $l = 1$, then $4j \leq 2q + 2k - 2 \leq 4j + 1$, so $k = 2j - q + 1$. There are $2j - 1 - q$ \mathcal{O}_{r_i} equal to \mathcal{O}_2 and 1 equal to \mathcal{O}_3 and 1 equal to \mathcal{O}_1 . When $l_1 = 1$, by (3.16) and (3.17), then $i_0 = j + 1 - \frac{q}{2}$ is unique. When $l_1 = 0$, then $i_0 = j + 2 - \frac{q}{2}$. So

$$IV = IV_a + IV_b + IV_c + IV_a^* + IV_b^* + IV_c^*, \tag{4.198}$$

where

$$IV_a = \sum_{0 \leq m_1 + m_2 \leq j - \frac{q}{2} - 1} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j - \frac{q}{2}} P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_1} (\mathcal{O}_1 \mathcal{L}_0^{-1})$$

$$\cdot (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_3^{-2} \mathcal{L}_0^{-1}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-1-m_1-m_2} I_{2j}; \quad (4.199)$$

$$\begin{aligned} \text{IV}_b = & \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_1} (\mathcal{O}_3^{-2} \mathcal{L}_0^{-1}) \\ & \cdot (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1 \mathcal{L}_0^{-1}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-1-m_1-m_2} I_{2j}; \end{aligned} \quad (4.200)$$

$$\begin{aligned} \text{IV}_c = & \sum_{0 \leq m_1 \leq j-\frac{q}{2}-1} \sum_{0 \leq m_2 \leq j-\frac{q}{2}} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1} P^N \\ & \cdot (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1 \mathcal{L}_0^{-1}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-m_2} I_{2j}. \end{aligned} \quad (4.201)$$

By (3.15) in [PZ], then

$$\mathcal{O}_3^{+2} = z_{j'} \frac{\partial \mathcal{R}_\cdot}{\partial z_{j'}}(0) + \bar{z}_{j'} \frac{\partial \mathcal{R}_\cdot}{\partial \bar{z}_{j'}}(0). \quad (4.202)$$

By (4.199), we have

$$\begin{aligned} \text{IV}_a = & \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N [(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \\ & \cdot (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1} \mathcal{O}_1) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} P^N]^* I_{2j}. \end{aligned} \quad (4.203)$$

Set

$$\begin{aligned} \text{IV}_a^1 = & \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N [(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \\ & \cdot (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1} \mathcal{O}_1') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} P^N]^* I_{2j}. \end{aligned} \quad (4.204)$$

$$\begin{aligned} \text{IV}_a^2 = & \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N [(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \\ & \cdot (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1} \mathcal{O}_1'') (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} P^N]^* I_{2j}. \end{aligned} \quad (4.205)$$

By (4.99)-(4.101) and direct computations, then

$$\left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \mathcal{L}_0^{-1} (z_{j'} \frac{\partial \mathcal{R}_\cdot}{\partial z_{j'}}(0)) A_2 \mathcal{P} \right] (0, Z) = 0, \quad (4.206)$$

$$\begin{aligned} & \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \mathcal{L}_0^{-1} (z_{j'} \frac{\partial \mathcal{R}_\cdot}{\partial z_{j'}}(0)) \mathcal{R}^{m_1+m_2} \right. \\ & \cdot \left. \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_1^{m_1,m_1+m_2} A_3 \mathcal{P} \right] (0, Z) \\ & = \frac{2}{(4\pi)^{j-\frac{q}{2}+1}} B_0^{1,m_1} B_1^{m_1,m_1+m_2} (B_0^{m_1+m_2+1,j-\frac{q}{2}} - B_1^{m_1+m_2+1,j-\frac{q}{2}}) \\ & \cdot (\mathcal{R}^\perp)^{j-\frac{q}{2}-1-m_1-m_2} \frac{\partial \mathcal{R}^\perp_\cdot}{\partial z_{j'}}(0) (\mathcal{R}^\perp)^{m_1+m_2} \left[\frac{2}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_{j'}}) \right. \\ & \left. + \frac{2}{3\pi} (\partial_{\bar{z}_{j'}} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi} (\partial_{j''} R^L)_{x_0} (\partial_{\bar{z}_{j'}}, e_{j''}) \right] \mathcal{P}(0, Z), \end{aligned} \quad (4.207)$$

$$\begin{aligned}
& \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \mathcal{L}_0^{-1} (z_{j'} \frac{\partial \mathcal{R}^\perp}{\partial z_{j'}}(0)) \right. \\
& \cdot (\mathcal{R}^\perp)^{m_1+m_2} \cdot \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_2^{m_1,m_1+m_2} A_4 \mathcal{P} \left. \right] (0, Z) \\
& = \frac{4}{3\pi \times (4\pi)^{j-\frac{q}{2}+1}} B_0^{1,m_1} B_2^{m_1,m_1+m_2} (B_2^{m_1+m_2+1,j-\frac{q}{2}} - B_1^{m_1+m_2+1,j-\frac{q}{2}}) \\
& \cdot \mathcal{R}^{j-\frac{q}{2}-1-m_1-m_2} \frac{\partial \mathcal{R}^\perp}{\partial z_{j'}}(0) (\mathcal{R}^\perp)^{m_1+m_2} \\
& \cdot [(\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_{j'}}) + (\partial_{\bar{z}_{j'}} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_r})] \mathcal{P}(0, Z), \tag{4.208}
\end{aligned}$$

$$\begin{aligned}
& \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \mathcal{L}_0^{-1} (\bar{z}_{j'} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j'}}(0)) (\mathcal{R}^\perp)^{m_1+m_2} \right. \\
& \cdot \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_0^{m_1,m_1+m_2} A_2 \mathcal{P} \left. \right] (0, Z) \\
& = \frac{1}{3\pi \times (4\pi)^{j-\frac{q}{2}+1}} B_0^{1,m_1} B_0^{m_1,m_1+m_2} (B_0^{m_1+m_2+1,j-\frac{q}{2}} - B_1^{m_1+m_2+1,j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^\perp)^{j-\frac{q}{2}-1-m_1-m_2} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j'}}(0) (\mathcal{R}^\perp)^{m_1+m_2} [4(\partial_{z_{j'}} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_r}) \\
& + 4(\partial_{z_r} R^L)_{x_0} (\partial_{z_{j'}}, \partial_{\bar{z}_r}) - (\partial_{j''} R^L)_{x_0} (\partial_{z_{j'}}, e_{j''})] \mathcal{P}(0, Z), \tag{4.209}
\end{aligned}$$

$$\begin{aligned}
& \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \mathcal{L}_0^{-1} (\bar{z}_{j'} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j'}}(0)) (\mathcal{R}^\perp)^{m_1+m_2} \right. \\
& \cdot \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_1^{m_1,m_1+m_2} A_3 \mathcal{P} \left. \right] (0, Z) \\
& = \frac{4}{3\pi \times (4\pi)^{j-\frac{q}{2}+1}} B_0^{1,m_1} B_1^{m_1,m_1+m_2} (B_2^{m_1+m_2+1,j-\frac{q}{2}} - B_1^{m_1+m_2+1,j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^\perp)^{j-\frac{q}{2}-1-m_1-m_2} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j'}}(0) (\mathcal{R}^\perp)^{m_1+m_2} \\
& \cdot [(\partial_{z_{j'}} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_r}) + (\partial_{z_r} R^L)_{x_0} (\partial_{z_{j'}}, \partial_{\bar{z}_r})], \tag{4.210}
\end{aligned}$$

$$\begin{aligned}
& \left[(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \mathcal{L}_0^{-1} (\bar{z}_{j'} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j'}}(0)) (\mathcal{R}^\perp)^{m_1+m_2} \right. \\
& \cdot \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_2^{m_1,m_1+m_2} A_4 \mathcal{P} \left. \right] (0, Z) = 0. \tag{4.211}
\end{aligned}$$

By (4.96), (4.202), (4.204) and (4.206)-(4.211), we get

$$\text{IV}_a^1 = \frac{-1}{2^{j-\frac{q}{2}} (j - \frac{q}{2})! (4\pi)^{2j-q+1} \times 3\pi} \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} B_0^{1,m_1} I_{2j}(\mathcal{R}^\perp)(x)^{j-\frac{q}{2}}$$

$$\begin{aligned}
& \times \left[(B_1^{m_1, m_1+m_2} B_0^{m_1+m_2+1, j-\frac{q}{2}} - B_1^{m_1, j-\frac{q}{2}} \right. \\
& + B_2^{m_1, j-\frac{q}{2}} - B_2^{m_1, m_1+m_2} B_1^{m_1+m_2+1, j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^{\perp, *}(x))^{m_1+m_2} (\nabla_{w_l}^{\wedge^{0, \bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp, *} (\mathcal{R}^{\perp, *}(x))^{j-\frac{q}{2}-1-m_1-m_2} (A_{r\bar{r}l} + A_{l\bar{r}r}) \\
& - \frac{1}{2} B_1^{m_1, m_1+m_2} (B_0^{m_1+m_2+1, j-\frac{q}{2}} - B_1^{m_1+m_2+1, j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^{\perp, *}(x))^{m_1+m_2} (\nabla_{w_l}^{\wedge^{0, \bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp, *} (\mathcal{R}^{\perp, *}(x))^{j-\frac{q}{2}-1-m_1-m_2} A_{e_r l e_r} \\
& + (B_1^{m_1, m_1+m_2} B_2^{m_1+m_2+1, j-\frac{q}{2}} - B_1^{m_1, j-\frac{q}{2}} \\
& + B_0^{m_1, j-\frac{q}{2}} - B_0^{m_1, m_1+m_2} B_1^{m_1+m_2+1, j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^{\perp, *}(x))^{m_1+m_2} (\nabla_{w_l}^{\wedge^{0, \bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp, *} (\mathcal{R}^{\perp, *}(x))^{j-\frac{q}{2}-1-m_1-m_2} (A_{l\bar{r}r} + A_{r\bar{l}r}) \\
& - \frac{1}{2} B_0^{m_1, m_1+m_2} (B_0^{m_1+m_2+1, j-\frac{q}{2}} - B_1^{m_1+m_2+1, j-\frac{q}{2}}) \\
& \left. \cdot (\mathcal{R}^{\perp, *}(x))^{m_1+m_2} (\nabla_{w_l}^{\wedge^{0, \bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp, *} (\mathcal{R}^{\perp, *}(x))^{j-\frac{q}{2}-1-m_1-m_2} A_{e_r \bar{l} e_r} \right] I_{2j}, \quad (4.212)
\end{aligned}$$

When $n = 4$, $q = 2$ and $j = 2$, we have

$$IV_a^2 = I_4(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) P^N [(\mathcal{L}_0^{-1} \mathcal{O}_3^{+2})(\mathcal{L}_0^{-1} P^{N, \perp} \mathcal{O}_1'') P^N]^* I_4. \quad (4.213)$$

By (4.202) and (4.158), we know that

$$IV_a^2 = \frac{\sqrt{-1}}{192\pi^3} \mathcal{R}^{\perp} I_{\det(\bar{W}^*) \otimes E} \left[\frac{1}{3} \hat{\Phi}^*(\partial_{z_r}) \frac{\partial \mathcal{R}^{0, *}}{\partial \bar{z}_r} + \frac{1}{2} \hat{\Phi}(\partial_{\bar{z}_r}) \frac{\partial \mathcal{R}^{0, *}}{\partial z_r} \right] I_4. \quad (4.214)$$

By (4.203)-(4.205), we get

$$IV_a = IV_a^1 + IV_a^2. \quad (4.215)$$

Nextly, we compute the term IV_b .

By (4.200), we have

$$\begin{aligned}
IV_b = & \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N [(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \\
& \cdot (\mathcal{L}_0^{-1} \mathcal{O}_1)(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2})(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} P^N]^* I_{2j}. \quad (4.216)
\end{aligned}$$

By (4.202), then

$$\begin{aligned}
& (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2})(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} \mathcal{P} \\
= & \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1, m_1+m_2+1} \mathcal{R}^{m_2} \frac{\partial \mathcal{R}_\cdot}{\partial z_{j'}}(0) \mathcal{R}^{m_1} z_{j'} \mathcal{P} \\
& + \frac{1}{2\pi \times (4\pi)^{m_1+m_2+1}} B_0^{1, m_1} B_1^{m_1+1, m_1+m_2+1} \mathcal{R}^{m_2} \frac{\partial \mathcal{R}_\cdot}{\partial \bar{z}_{j'}}(0) \mathcal{R}^{m_1} b_{j'} \mathcal{P} \\
& + \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1, m_1+m_2+1} \mathcal{R}^{m_2} \frac{\partial \mathcal{R}_\cdot}{\partial \bar{z}_{j'}}(0) \mathcal{R}^{m_1} \bar{z}_{j'} \mathcal{P}. \quad (4.217)
\end{aligned}$$

By (4.97) and (4.217) and direct computations, we know that after integration

$$\begin{aligned}
& (A_1 + A_2 + A_3 + A_4)(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_2}(\mathcal{L}_0^{-1}\mathcal{O}_3^{+2})(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_1}\mathcal{P} \\
= & \frac{1}{2\pi \times (4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} (\mathcal{R}^\perp)^{m_2} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j''}}(0) (\mathcal{R}^\perp)^{m_1} (b_{j''} z_{j'} + 2\delta_{j'j''}) \mathcal{P} \\
& \times \left[\frac{4}{3} (\partial_{z_{j'}} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_\alpha}) + \frac{4}{3} (\partial_{z_\alpha} R^L)_{x_0} (\partial_{z_{j'}}, \partial_{\bar{z}_\alpha}) - \frac{1}{3} (\partial_\alpha R^L)_{x_0} (\partial_{z_{j'}}, e_\alpha) \right] \\
& - \frac{1}{3\pi \times (4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} (\mathcal{R}^\perp)^{m_2} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j'}}(0) (\mathcal{R}^\perp)^{m_1} \\
& \cdot (\partial_{z_\beta} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_l}) (2\delta_{lj'} b_\alpha z_\beta + 4\delta_{lj'} \delta_{\alpha\beta}) \mathcal{P} \\
& + \frac{1}{(4\pi)^{m_1+m_2+1}} B_0^{1,m_1+m_2+1} (\mathcal{R}^\perp)^{m_2} \frac{\partial \mathcal{R}^\perp}{\partial z_{j'}}(0) (\mathcal{R}^\perp)^{m_1} b_\alpha z_{j'} \mathcal{P} \\
& \times \left[\frac{2}{3\pi} (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_\alpha}) + \frac{2}{3\pi} (\partial_{\bar{z}_\alpha} R^L)_{x_0} (\partial_{z_r}, \partial_{\bar{z}_r}) - \frac{1}{6\pi} (\partial_{j''} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, e_{j''}) \right] \\
& + \frac{1}{3\pi \times (4\pi)^{m_1+m_2+1}} B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} (\mathcal{R}^\perp)^{m_2} \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_{j'}}(0) (\mathcal{R}^\perp)^{m_1} \\
& \cdot (\partial_{z_{j''}} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) (b_l b_{j'} z_\alpha z_{j''} + 2\delta_{\alpha j'} b_l z_{j''} + 2\delta_{j'j''} b_l z_\alpha) \mathcal{P} \\
& + \frac{1}{3\pi \times (4\pi)^{m_1+m_2+1}} B_0^{1,m_1+m_2+1} (\mathcal{R}^\perp)^{m_2} \frac{\partial \mathcal{R}^\perp}{\partial z_{j'}}(0) (\mathcal{R}^\perp)^{m_1} \\
& \cdot (\partial_{\bar{z}_r} R^L)_{x_0} (\partial_{z_\alpha}, \partial_{\bar{z}_l}) b_l b_r z_\alpha z_{j'} \mathcal{P}
\end{aligned} \tag{4.218}$$

Set

$$\begin{aligned}
\text{IV}_b^1 &= \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N [(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \\
& \cdot (\mathcal{L}_0^{-1}(A_1 + A_2 + A_3 + A_4)) (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1}\mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_1} P^N]^* I_{2j}. \tag{4.219}
\end{aligned}$$

$$\begin{aligned}
\text{IV}_b^2 &= \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N [(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \\
& \cdot (\mathcal{L}_0^{-1}A_5) (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1}\mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_1} P^N]^* I_{2j}. \tag{4.220}
\end{aligned}$$

$$\begin{aligned}
\text{IV}_b^3 &= \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}} P^N [(\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1-m_2} \\
& \cdot (\mathcal{L}_0^{-1}\mathcal{O}_1'') (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_2} (\mathcal{L}_0^{-1}\mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1}\mathcal{O}_2^{+2})^{m_1} P^N]^* I_{2j}. \tag{4.221}
\end{aligned}$$

Then

$$IV_b = \text{IV}_b^1 + \text{IV}_b^2 + \text{IV}_b^3. \tag{4.222}$$

By (4.218), we get

$$\text{IV}_b^1 = \frac{1}{2^{j-\frac{q}{2}} (j - \frac{q}{2})! (4\pi)^{2j-q+1} \times 3\pi} \sum_{0 \leq m_1+m_2 \leq j-\frac{q}{2}-1} I_{2j} (\mathcal{R}^\perp)(x)^{j-\frac{q}{2}}$$

$$\begin{aligned}
& \cdot I_{\det(\bar{W}^*) \otimes E} \left[(-2B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_0^{m_1+m_2+1,j-\frac{q}{2}} \right. \\
& + 2B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_1^{m_1+m_2+1,j-\frac{q}{2}} \\
& - B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_2^{m_1+m_2+1,j-\frac{q}{2}} + B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_1^{m_1+m_2+1,j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^{\perp,*}(x))^{m_1} (\nabla_{\bar{w}_l}^{\wedge^{0,\bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp,*} (\mathcal{R}^{\perp,*}(x))^{j-\frac{q}{2}-1-m_1} A_{\bar{r}l\bar{r}} \\
& - (B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_0^{m_1+m_2+1,j-\frac{q}{2}} - B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_1^{m_1+m_2+1,j-\frac{q}{2}} \\
& + B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_2^{m_1+m_2+1,j-\frac{q}{2}} - B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} B_1^{m_1+m_2+1,j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^{\perp,*}(x))^{m_1} (\nabla_{\bar{w}_l}^{\wedge^{0,\bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp,*} (\mathcal{R}^{\perp,*}(x))^{j-\frac{q}{2}-1-m_1} A_{\bar{l}\bar{r}r} \\
& + (B_0^{1,m_1+m_2+1} B_1^{m_1+m_2+1,j-\frac{q}{2}} - B_0^{1,m_1+m_2+1} B_2^{m_1+m_2+1,j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^{\perp,*}(x))^{m_1} (\nabla_{\bar{w}_l}^{\wedge^{0,\bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp,*} (\mathcal{R}^{\perp,*}(x))^{j-\frac{q}{2}-1-m_1} (A_{r\bar{r}l} + A_{\bar{l}\bar{r}r}) \\
& + \frac{1}{2} B_0^{1,m_1} B_1^{m_1+1,m_1+m_2+1} (B_0^{m_1+m_2+1,j} - B_1^{m_1+m_2+1,j-\frac{q}{2}}) \\
& \cdot (\mathcal{R}^{\perp,*}(x))^{m_1} (\nabla_{\bar{w}_l}^{\wedge^{0,\bullet} \otimes E}(\mathcal{R}_\cdot))^{\perp,*} (\mathcal{R}^{\perp,*}(x))^{j-\frac{q}{2}-1-m_1} A_{e_r \bar{l} e_r} \\
& - \frac{1}{2} B_0^{1,m_1+m_2+1} B_1^{m_1+m_2+1,j-\frac{q}{2}} (\mathcal{R}^{\perp,*}(x))^{m_1} \\
& \cdot (\nabla_{\bar{w}_l}^{\wedge^{0,\bullet} \otimes E}(\mathcal{R}_\cdot))^* (\mathcal{R}^{\perp,*}(x))^{j-\frac{q}{2}-1-m_1} A_{e_r l e_r} \Big], \tag{4.223}
\end{aligned}$$

By (4.220), when $n = 4$, $q = 2$ and $j = 2$

$$IV_b^2 = I_4(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2}) P^N [(\mathcal{L}_0^{-1} A_5)(\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) P^N]^* I_4. \tag{4.224}$$

Then (4.102) and (4.202), we obtain

$$\begin{aligned}
IV_b^2 &= \frac{1}{8\pi^4} \mathcal{R}^{\perp} I_{\det(\bar{W}^*) \otimes E} \left[\frac{-1}{144} (\partial_{z_r} R^L)_{x_0} (\partial_{\bar{z}_\alpha}, \partial_{z_\alpha}) \frac{\partial \mathcal{R}^{\perp,*}}{\partial \bar{z}_r} \right. \\
& \quad \left. + \frac{1}{192} (\partial_{j'} R^L)_{x_0} (\partial_{z_r}, e_{j'}) \frac{\partial \mathcal{R}^{\perp,*}}{\partial \bar{z}_r} \right] I_4. \tag{4.225}
\end{aligned}$$

When $n = 4$, $q = 2$ and $j = 2$,

$$IV_b^3 = 0. \tag{4.226}$$

Set

$$\begin{aligned}
IV_c^1 &= \sum_{0 \leq m_1 \leq j-\frac{q}{2}-1} \sum_{0 \leq m_2 \leq j-\frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1} P^N \\
& \quad \cdot (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1' \mathcal{L}_0^{-1}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-m_2} I_{2j}. \tag{4.227}
\end{aligned}$$

$$\begin{aligned}
IV_c^2 &= \sum_{0 \leq m_1 \leq j-\frac{q}{2}-1} \sum_{0 \leq m_2 \leq j-\frac{q}{2}} I_{2j}(\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{m_1} (\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2})^{j-\frac{q}{2}-1-m_1} P^N \\
& \quad \cdot (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{m_2} (\mathcal{O}_1'' \mathcal{L}_0^{-1}) (\mathcal{O}_2^{-2} \mathcal{L}_0^{-1})^{j-\frac{q}{2}-m_2} I_{2j}. \tag{4.228}
\end{aligned}$$

Then

$$IV_c = IV_c^1 + IV_c^2. \quad (4.229)$$

By (4.24), (4.187) and (4.217), we get

$$\begin{aligned} IV_c^1 = & \sum_{0 \leq m_1 \leq j - \frac{q}{2} - 1} \sum_{0 \leq m_2 \leq j - \frac{q}{2}} \frac{1}{3\pi \times (4\pi)^{2j-q+1}} \\ & \cdot (B_0^{1, j - \frac{q}{2}} - B_0^{1, j - \frac{q}{2} - m_1 - 1} B_1^{j - \frac{q}{2} - m_1, j - \frac{q}{2}})(\mathcal{R}^\perp)(x)^{m_1} \\ & \cdot (\nabla_{\bar{w}_l}^{\wedge 0, \bullet} \otimes E(\mathcal{R}_\cdot))^\perp (\mathcal{R}^\perp)(x)^{j - \frac{q}{2} - 1 - m_1} I_{\det(\bar{W}^*) \otimes E}(\mathcal{R}^{\perp, *}(x))^{j - \frac{q}{2}} \\ & \cdot \left[(B_0^{1, j - \frac{q}{2} - m_2} B_0^{j - \frac{q}{2} - m_2, j - \frac{q}{2}} - 2B_0^{1, j - \frac{q}{2} - m_2} B_1^{j - \frac{q}{2} - m_2, j - \frac{q}{2}} \right. \\ & \quad \left. + B_0^{1, j - \frac{q}{2} - m_2} B_2^{j - \frac{q}{2} - m_2, j - \frac{q}{2}}) A_{\bar{l}_r \bar{r}} \right. \\ & \quad \left. - (B_0^{1, j - \frac{q}{2} - m_2} B_1^{j - \frac{q}{2} - m_2, j - \frac{q}{2}} - B_0^{1, j - \frac{q}{2} - m_2} B_2^{j - \frac{q}{2} - m_2, j - \frac{q}{2}}) A_{\bar{r} r \bar{l}} \right. \\ & \quad \left. - \frac{1}{2} (B_0^{1, j - \frac{q}{2} - m_2} B_0^{j - \frac{q}{2} - m_2, j - \frac{q}{2}} - B_0^{1, j - \frac{q}{2} - m_2} B_1^{j - m_2, j - \frac{q}{2}}) A_{e_r \bar{l} e_r} \right], \end{aligned} \quad (4.230)$$

By (4.228), when $n = 4$, $j = 2$ and $q = 2$, one has

$$IV_c^2 = I_4(\mathcal{L}_0^{-1} \mathcal{O}_3^{+2}) P^N (\mathcal{L}_0^{-1} \mathcal{O}_2^{+2} \mathcal{L}_0^{-1} \mathcal{O}_1'' P^N)^* I_4. \quad (4.231)$$

Similar to IV_b^2 , we obtain

$$IV_c^2 = \frac{5\sqrt{-1}}{24 \times 144\pi^3} I_4 \frac{\partial \mathcal{R}^\perp}{\partial \bar{z}_r} I_{\det(\bar{W}^*) \otimes E} \hat{\Phi}^*(\partial_{z_r}) \mathcal{R}^{0, *} I_4. \quad (4.232)$$

As in [PZ, p.20], we may write the above formulas in the intrinsic way. When $n = 4$, $j = 2$ and $q = 2$, by (4.1), (4.3), (4.4)-(4.8), (4.11), (4.34), (4.37), (4.76), (4.79)-(4.83), (4.86), (4.151)-(4.153), (4.159), (4.162)-(4.172), (4.182), (4.183), (4.189), (4.194)-(4.198), (4.212), (4.214), (4.215), (4.222), (4.223), (4.225), (4.226), (4.229), (4.230), (4.232), we prove Theorem 1.4. \square

Acknowledgements. This work was supported by NSFC No.11271062 and NCET-13-0721. The author is indebted to referees for their careful reading and helpful comments.

References

- [Bi]Bismut, J.: A local index theorem for non Kähler manifolds. Math. Ann. 284, 681-699 (1989)
- [Ca]Catlin, D.: The Bergman kernel and a theorem of Tian. In: Analysis and Geometry in Several Complex Variables, Katata, 1997. Trends Math., pp. 1-23. Birkhauser, Boston (1999)
- [CS]Charbonneau B., Stern M.: Asymptotic Hodge theory of vector bundles. arXiv: 1111.0591 (2011)

- [DLM]Dai, X., Liu, K., Ma, X.: On the asymptotic expansion of Bergman kernel. J. Differ. Geom. 72, 1-41 (2006). Announced in C. R. Math. Acad. Sci. Paris 339, 193-198 (2004)
- [LuW1]Lu, W.: The second coefficient of the asymptotic expansion of the Bergman kernel of the Hodge-Dolbeault operator. J. Geom. Anal., DOI 10.1007/s12220-013-9412-y.
- [LuW2]Lu, W.: Morse Inequalities and Bergman Kernels. Doctor thesis, (2013)
- [LuZ]Lu, Z.: On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch. Am. J. Math. 122, 235-273 (2000)
- [MM06]Ma, X., Marinescu, G.: The first coefficients of the asymptotic expansion of the Bergman kernel of the Spinc Dirac operator. Int. J. Math. 17, 737-759 (2006)
- [MM07]Ma, X., Marinescu, G.: Holomorphic Morse Inequalities and Bergman Kernels. Progr. Math., vol. 254. Birkhauser, Basel (2007)
- [MM08]Ma, X., Marinescu, G.: Generalized Bergman kernels on symplectic manifolds. Adv. Math. 217, 1756-1815 (2008)
- [MM12]Ma, X., Marinescu, G.: Berezin-Toeplitz quantization on Kähler manifolds. J. Reine Angew. Math. 662, 1-56 (2012)
- [PZ]Puchol, M., Zhu, J.: The first terms in the expansion of the Bergman kernel in higher degrees. arXiv:1210.1717 (2012)
- [Ru]Ruan, W.: Canonical coordinates and Bergman metrics. Commun. Anal. Geom. 6, 589-631 (1998)
- [Ti]Tian, G.: On a set of polarized Kähler metrics on algebraic manifolds. J. Differ. Geom. 32, 99-130 (1990)
- [Wa]Wang, X.: Canonical metrics on stable vector bundles. Commun. Anal. Geom. 13, 253-285 (2005)
- [Ze]Zelditch, S.: Szegő kernels and a theorem of Tian. Int. Math. Res. Not. 6, 317-331 (1998)

School of Mathematics and Statistics, Northeast Normal University, Changchun Jilin, 130024, China

E-mail: *wangy581@nenu.edu.cn*